The Free Group : an introduction

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Definition

*A **group** G is a set equipped with an operation " $\cdot$ " satisfying*

1. *associativity*
2. *identity*
3. *inverse*

Exemples

The trivial group $\{e\}$ and $(\mathbb{Z}, +)$ are groups, but not $(\mathbb{N}, +)$.

Concatenation

Definition

Let X be a set and consider $\overline{X} = X \cup \{x^{-1} : x \in X\}$ with the concatenation law : $x \cdot y = xy$.

Example :

"Euler" is the concatenation of "E,u,l,e,r".

The set of words

Definition

We define ***the set of words on*** X *as* $\mathcal{M}(X) = \{s_1 s_2 \cdots s_n : s_i \in \overline{X}\} \cup \emptyset$*, where* $\emptyset$ *is the empty word.*

Equivalence relation

Definition

An equivalence relation $\sim$ *is a relation that is* ***transitive, symmetric, reflexive.*** *The equivalence classe of* x *is the set* $[x] := \{y : y \sim x\}$.

Equivalence between words

We say that two words are **equivalent** if and only if one can be transformed into the other by inserting or deleting a finite number of "xx^{-1}".

The free group on X

Definition

*The **free group on** X is the collection of all such equivalence classes on $\mathcal{M}(X)$ and is denoted by $\mathcal{F}(X)$.*

Proposition

The free group is a group

Proof.

$\emptyset$ is the identity, concatenation is the law, and since $s_i s_i^{-1} = s_i^{-1} s_i = \emptyset$, the inverse of $s_1 \cdots s_n$ is $s_n^{-1} \cdots s_1^{-1}$. □

Examples

1. The free group on the empty set contains an unique element : the empty word.
2. The free group on a singleton $\{x\}$ is

$$\mathcal{F}(\{x\}) = \{x^k : k \in \mathbb{Z}\} \simeq \mathbb{Z}$$

3. **But**, $\mathcal{F}(\{x, y\})$ is not isomorphic to $\mathbb{Z}^2$.

Subgroup

Definition

We say that H is a subgroup of G if $H \subset G$ and if it is a group under the law of G.

Examples :

1. $\{e_G\}$ is always a subgroup of G.
2. G is a subgroup of itself.
3. The set $n\mathbb{Z}$ of all multiples of $n \in \mathbb{Z}$ is a subgroup of $\mathbb{Z}$, and every subgroup of $(\mathbb{Z}, +)$ is of this form.

Properties of $\mathcal{F}$

$\mathcal{F}(a, b) \leq \mathcal{F}(a, b, c) \leq \mathcal{F}(a, b)$

Proposition

$\mathcal{F}(a, b)$ et $\mathcal{F}(a, b, c)$ are mutual subgroups.

- $\mathcal{F}(a, b) \leq \mathcal{F}(a, b, c)$ is a consequence of $\{a, b\} \subset \{a, b, c\}$
- The inverse inequality is obtained by proving that $\mathcal{F}(a, b, c) \simeq \langle a^2, b^2, ab \rangle$

The homophonic group

Definition

Let $A = \{a, b, c, ..., z\}$*. We define the* ***homophonic group*** *on* A *as* $H = \mathcal{F}(A)/\sim$ *with*

$$x \simeq y \iff x \text{ and } y \text{ are homophonic}$$

H is trivial

Proposition

The homophonic group is trivial.

It suffices to have a sufficiently long list of homophones to conclude that $a = b = c = d = e = ... = y = z = \emptyset$. As an example, $e = \emptyset$, since $bee = be$.