# Invariants in Knot Theory

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**Abstract**

In this expository article, we introduce the basics of knot theory. We then discuss several invariants appearing in knot theory including linking number, tricolorability, the bracket polynomial, and the Jones polynomial.

## 1 Knot Theory

In this expository article largely [Ada94], we introduce the basics of knot theory. In Section 1 we define knots, knot projections, and introduce Reidmeister moves. In Section 2 we define what an invariant is then discuss several invariants appearing in knot theory including linking number, tricolorability, the bracket polynomial, and the Jones polynomial.

### 1.1 Introduction

**Definition 1.2.** *A* ***knot*** *is a knotted loop of string, except that we think of the string as having no thickness, its cross-section being a single point. The knot is then a closed curve in space that does not intersect itself anywhere.*

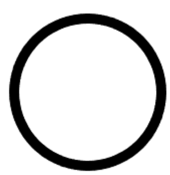


Figure 1: The unknot

**Definition 1.3.** *A* ***knot projection*** *is a picture of the knot but in 2D where it is flattened and each crossing is shown by layering the knot on different levels. The projections below represent single-crossing knots which are knots that contain only one crossing.*

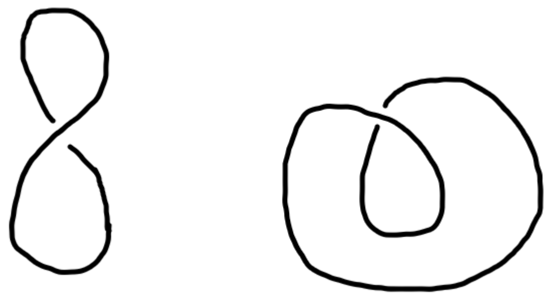


Figure 2: Projections of the unknot

## 1.4 Reidmeister Moves

**Definition 1.5.** *Planar isotopy is a deformation of a knot projection if it deforms the projection plane as if it were made of rubber with the projection drawn upon it.*

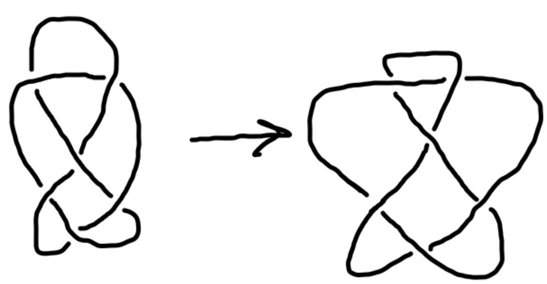


Figure 3: Example of planar isotopy

**Example 1.6.**

However, there are obviously some ways to change the crossings of a knot. Any possible change in crossings is one of three distinct categories, called Reidemeister moves. A type I Reidemeister move allows us to create or destoy a twist in the knot as shown. A type II Reidemeister move adds or removes two crossings as seen in this example. The final Reidemeister move, the type III, slides a strand of a knot over a crossing.

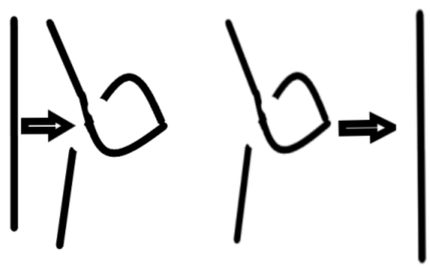


Figure 4: Reidmeister Move 1

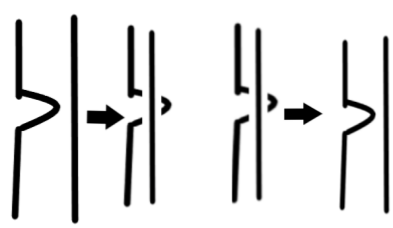


Figure 5: Reidmeister Move II

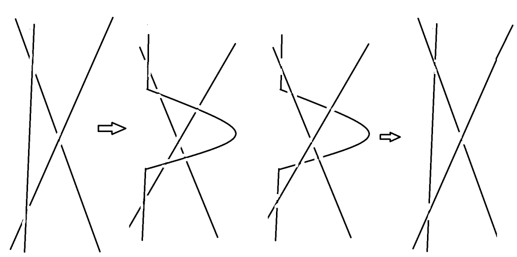


Figure 6: Reidemeister Move III

# 2 Invariants

## 2.1 Introduction to Invariants

In mathematics, an **invariant** is a property of an object that isn't changed by certain types of transformations. In knot theory, these objects are knot projections, and the transformations are Reidemeister moves and planar isotopy.

## 2.2 Linking number

A **link** is combination of multiple knots. To orient a link, simply assign a direction to the string of each knot. The **linking number** of a link is an invariant we can use to distinguish between oriented links. It is found by assigning a value of 1 or $-1$ to crossings involving both knots, then adding them up. Any possible crossing can be found by rotating one of these diagrams.

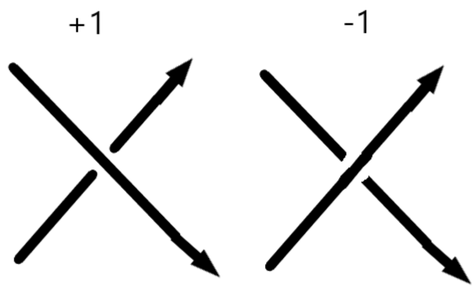



Figure 7: Linking number associated to each crossing

We can quickly prove that this is an invariant by checking to see if any of the Reidemeister moves, the only things that change can change crossings without changing the link, have an effect on the linking number.

**Proposition 2.3.** *Linking number is an invariant of knot diagrams.*

*Proof.* Reidemeister I only involves one knot, so we cant automatically rule it out. As for Reidemeister II, the two crossings created or destroyed by it will always have opposite linking numbers, as shown in the example. As for Reidmeister III, the crossings created have the same net value as the crossings destroyed.

□

## 2.4 Tricolorability

We have have now proved an invariant of link diagrams, but what about an invariant that distinguishes knots? The first step to finding this invariant, called tricolorability, is to identify each part of the knot that runs from one under-crossing to the next under-crossing as a separate strand. We the assign on of three colors to each strand in the knot. Now we say that a knot is **tricolorable** if it's crossings involve only 1 or 3 different colors. This will allow us to prove that, for example, the unknot cant be tangled into the above knot, because with no crossings, the unknot can't be tricolorable. However, we still have to run through the Reidemeister moves to prove that tricolorabilty is an invariant.

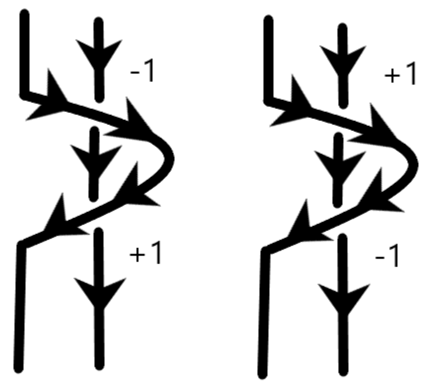



Figure 8: Reidemeister II move preserves linking number

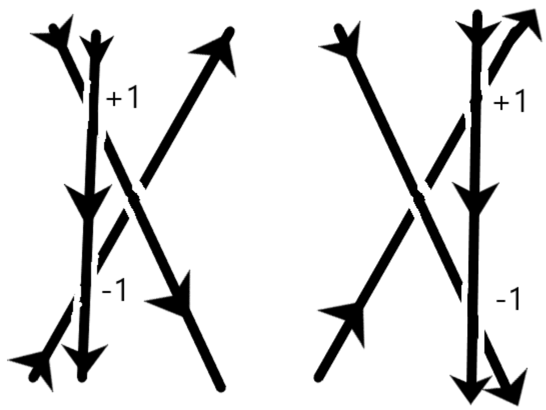



Figure 9: Reidemeister III move preserves linking number

**Proposition 2.5.** *Tricolorabilty is an invariant of knot diagrams.*

*Proof.* When we twist or untwist the knot in a Reidemeister I move, we can leave all the strands the same color, thus preserving tricolorability. With a Reidemeister II, either all the strands are the same color, or the crossings created or destroyed have 3 colors. Either way, tricolorabilty is unaffected. Finally, as with the previous moves, if the strands involved in a Reidemeister III move are all same color, it won't affect tricolorabilty. Even if they aren't, the move still preserves tricolorability. □

Since the trfoil is tricolorable but the unknot isn't, tricolorability distinguishes these two knots as different knots. In other words, the trefoil cannot be untangled into the unknot.

Figure 10: A tricolorable projection of the trefoil

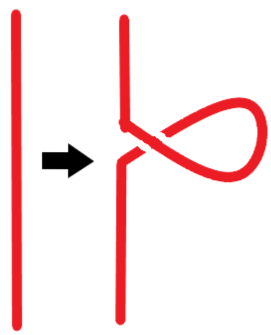


Figure 11: Reidemeister I move preserves tricolorability

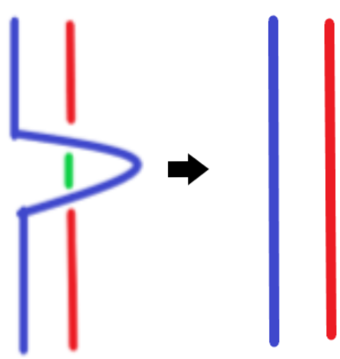


Figure 12: Reidemeister II move preserves tricolorability

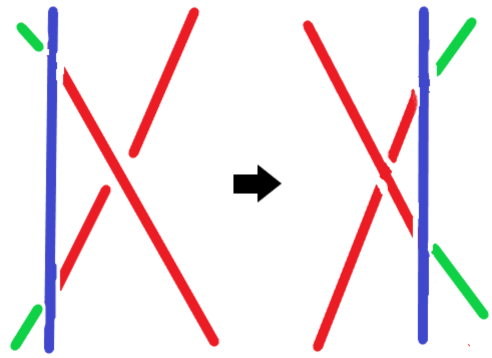


Figure 13: Reidemeister III move preserves tricolorability

## 2.6 Definition of Polynomial

**Definition 2.7.** *A* ***polynomial*** *is an expression consisting of variables and coefficients, that involves only the operations of addition, subtraction, multiplication, and non-negative integer exponentiation of variables. There are different types of polynomials as well such as the* ***Laurent Polynomial****, which can have both positive and negative powers of $t$.*

## 2.8 Bracket Polynomial

**Definition 2.9.** *A* ***bracket polynomial*** *is a polynomial developed for knots by mathematicians. It isn't an invariant of knots but it is still important.*

Here's rule 1.

$$\langle \circ \rangle = 1$$

Rule 2 is given by the following.Given a crossing in our link projection, we split it open vertically and horizontally, in order to obtain two new link projections, each of which has one fewer crossing. We make the bracket polynomial of our own link projection a linear combination of the bracket polynomials of our two new link projections, where we have decided the coefficients to be $A$ and $B$.

Rule 3 is given by the following. Finally we would like a rule for adding in a trivial component to a link (the result of which will always be a split link).

## 2.10 Bracket Polynomial and Reidemeister Moves

For bracket polynomials, for them to be an invariant of knots, they must be unchanged by Reidemeister moves. They remain the same for Reidemeister II adn II moves, but are changed by Reidemeister I. For example, here is a proof showing that Reidemiester II moves don't change the bracket polynomial.

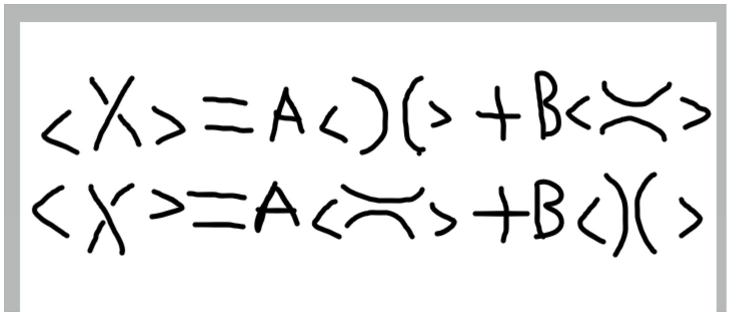


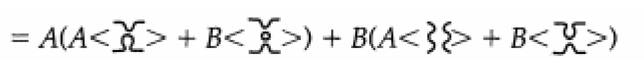
Figure 14: Bracket Polynomial Rule 2

**Proposition 2.11.** *Reidemiester II preserves the bracket polynomial.*

*Proof.* To do that, here's what we need to prove:

Now we can use rule 2 on top crossing of the left polynomial.

Next we use rule 2 on the bottom crossing and add the result to the top crossing.



We can then use rule 3 on the bracket with a circle in it.

Now we can just simplify.

To make these two expressions equal, we need to make $B = A^{-}1$ and C $= -A^2 - A^{-}2$. This changes our rules, but not fundamentally. □

## 2.12 Writhe

Writhe is really just the same concept as linking numbers, except they apply to one-part knots as well. That means our Reidemeister II and III proofs still apply, but Reidemeister I moves change the writhe of knot, meaning its not an invariant. It can still be used for our polynomial knot invariant, however.

## 2.13 Jones Polynoimal

This polynomial knot invariant is called the Jones polynomial, and it will remain unchanged for all 3 Reidemeister moves. This will use the bracket polynomial of a knot, which we will call $\langle K \rangle$, so we start out with $\langle K \rangle$. But we must add a new rule to make it an invariant, so we will multiply something by $\langle K \rangle$. It will involve the writhe of the same polynomial which we write as $w(L)$, This new rule is going to be $(-A^3)^{-w(L)} * \langle K \rangle$.

## 2.14 Reidemeister moves

Both writhes and bracket polynomials are unchanged by Reidemeister moves II and III, so we only have to prove that the Jones polynomial remains the same for Reidemeister I moves.

**Proposition 2.15.** *Reidemeister I preserves the Jones polynomial.*

*Proof.* To show this, we have the Jones polynomial of $K'$, the same knot but with a Reidemeister I move that increases the writhe by 1.

$$(-A^3)^{-w(K')} * \langle K' \rangle$$

Because this move increases the writhe by 1, we can rewrite $w(K')$ as $w(K)+1$. Reidmeister I moves multiply the bracket polynomial of a knot by $(-A)^3$, so we can rewrite $\langle K' \rangle$ as $(-A)^3 * \langle K \rangle$.

$$(-A^3)^{-(w(K)+1)} * (-A)^3 * \langle K \rangle$$

Now we can distribute the negative sign at the beginning of $-(w(K)+1)$, giving us $-w(K) - 1$.

$$(-A^3)^{-w(K)-1} * (-A)^3 * \langle K \rangle$$

We can now use exponent rules to rewrite $(-A^3)^{-w(K)-1}$ as $(-A^3)^{-w(K)} * (-A^3)^{-1}$.

$$(-A^3)^{-w(K)} * (-A^3)^{-1} * (-A)^3 \langle K \rangle$$

$(-A)^3$ is equal to $(-A^3)^1$, therefore can now use exponent rules to add them together, getting $(-A^3)^0 = 1$.

$$(-A^3)^{-w(K)} * 1 * \langle K \rangle$$

Multiplying by 1 doesn't affect the equation, so we end up back where we started. □

This proves that the Jones polynomial is a knot invariant. But there is one more step to turn this equation into the Jones Polynomial: replace A with $t^{-\frac{1}{4}}$. This is the final version of the Jones polynomial.

**Definition 2.16.** *The Jones polynomial of a knot diagram is given by*

$$(-(t^{-\frac{1}{4}})^3)^{-w(K)} * \langle K \rangle$$

## 2.17 Applications

We now have an invariant that works for all knots. This allows us prove that plenty of pairs knots are distinct, but it isn't known whether the Jones polynomial distinguishes all knots from the unknot. However this has been proven for each possible knot with at most 24 crossings [TS21].

# References


[Ada94] Colin Adams, *The knot book: An elementary introduction to the mathematical theory of knots*, W. H. Freeman, New York, 1994.

[TS21] Robert E Tuzun and Adam S Sikora, *Verification of the jones unknot conjecture up to 24 crossings*, Journal of Knot Theory and Its Ramifications **30** (2021), no. 03, 2150020.