NUMBERS, COLORS, AND KNOTS

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Abstract. This paper is about the knot theory. We focus mainly on tricolorability, Dowker's notation, and knots and sticks. First, we will review the basic definition of knots and give some examples. Later, we will define tricolorability, an invariant for knots, and Dowker's notation, a way to encode the projection of a knot using numbers. Finally, we will talk about some interesting formulas regarding constructing a knot with sticks.

1. Introduction

First, we recall the basic definitions of knots. A knot is a loop in space that does not intersect itself anywhere. Knots can be trivial or nontrivial, and in particular, when one knot is trivial and another knot is not, then these knots are different. Second, we talk about tricolorability. Tricolorability is another method to differentiate knots. This is about coloring the knot with three colors according to some rules. Some knots are tricolorable and others are not tricolorable. This means that these knots are different. Later, we cover Dowker's notation. Dowker's notation is a method that associates a knot with an array of numbers. This is useful because the numbers are more familiar objects in math. Sometimes, given one of these arrays of numbers, we can recover a knot from it. Finally, we talk about knots and sticks, which is about creating knots using sticks (or you can use your body to create a knot using your arms like sticks), and a formula that tells you an upper bound for how many sticks you will need to create a knot with a given number of crossings.

2. Definition of Knots

Definition 2.1. A *knot* is a loop in space that does not intersect itself anywhere.

We say that two knots are equal when each of the knots can be untied and wrapped into the other knot. We call this an *isotopy*.

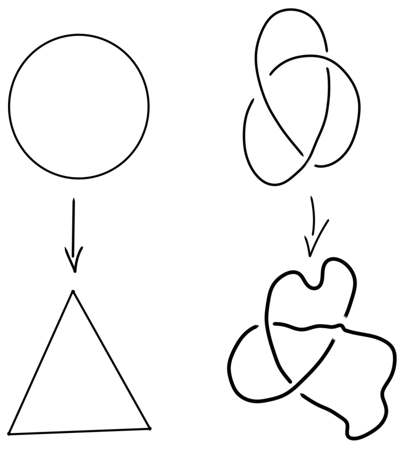


Figure 1. Examples of isotopies

We always work with *projections* of knots, which are curves on a flat surface representing the knots. In contrast with knots, their projections can (and most of the times) self-intersect. Since the knots are objects of the third dimension, we use the projections to be able to show how these knots look in a space of the second dimension.

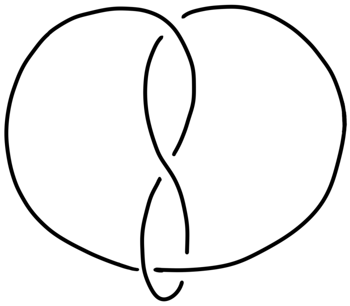


FIGURE 2. The figure-eight knot

Let's give some examples of knots. The one that is just a circle is the unknot. You can transform or tangle this knot in many shapes, but if you untie it, it will be the same unknot. This knot can be also called the *trivial knot*.

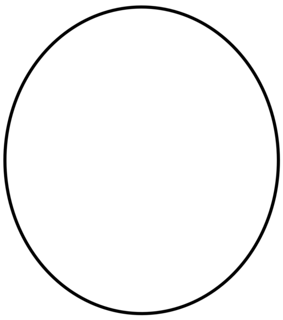


FIGURE 3. Unknot

The second type of knot are the nontrivial knots. These are the knots which we cannot untie. It turns out that any projection of these knots must always have three or more crossings.

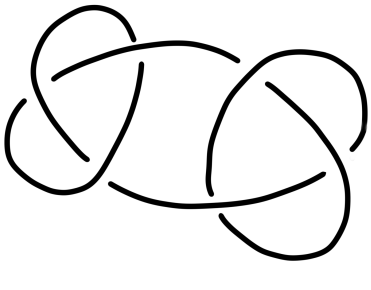
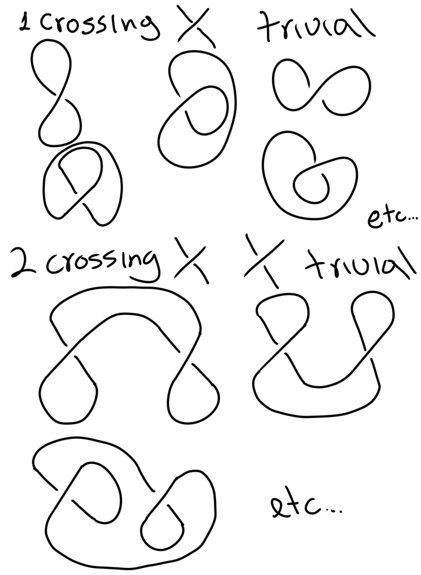


FIGURE 4. Double trefoil

We can check all the cases for one crossing and two crossing and they all turn out to be trivial knots, so nontrivial knots must have at least three crossings.



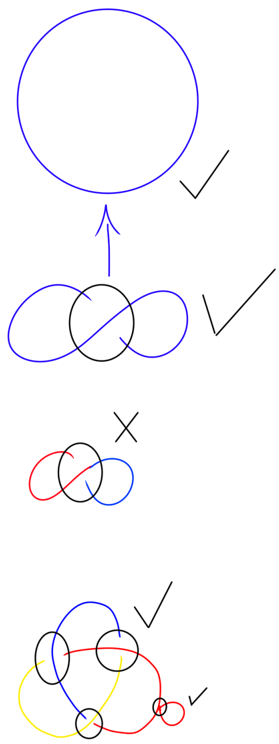
In knot theory, we are interested in knowing when two knots are different since this is usually difficult to see.

3. Tricolorability

Tricolorability is another method to differentiate knots. This is an invariant that consists of coloring the knot with three colors according to some rules. The power of this method is that some knots are tricolorable and others are not. Thus, if a knot is tricolorable and other knot is not, these knots are different. The rules of tricolorability are the following:

- We use at least 2 colors.
- We color each arc of the projection of the knot with one of three colors.
- Each crossing must have either 1 color or 3 colors, but it cannot have only 2 colors.

Let us see an example about the rules.



Here is another example about tricolorability. We will use the 7_4 knot. First, we will show a picture of this knot without any color, and then we will show you the version with tricolorability.

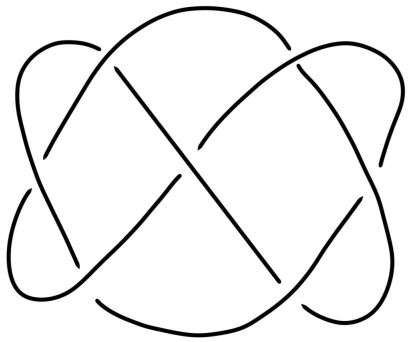


FIGURE 5. 7_4 knot

We will use tricolorability in the 7_4 knot. We will show how this 7_4 knot would look with tricolorability following the rules that we explained above.

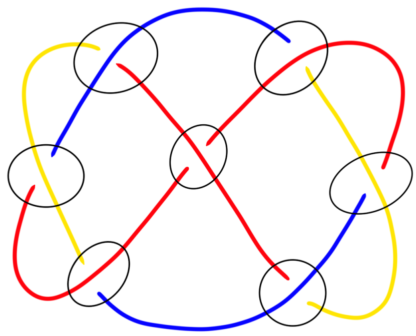


FIGURE 6. 7_4 knot

Thus, we get that the 7_4 knot is tricolorable. This is good. In particular, as 7_4 is tricolorable and we know that the unknot is not tricolorable, the 7_4 knot is indeed distinct from the unknot. As we explained above, when one knot is tricolorable and another is not, this mean that these knots are different.

4. Dowker's Notation

Dowker's notation is a method we use to encode a knot using numbers. We can also use it to produce knots in some situations. We use it because it is more common in math to identify things with numbers. We will give you an example with the trefoil knot.

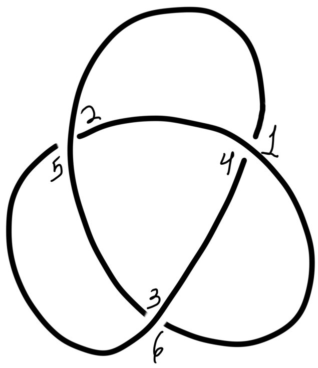


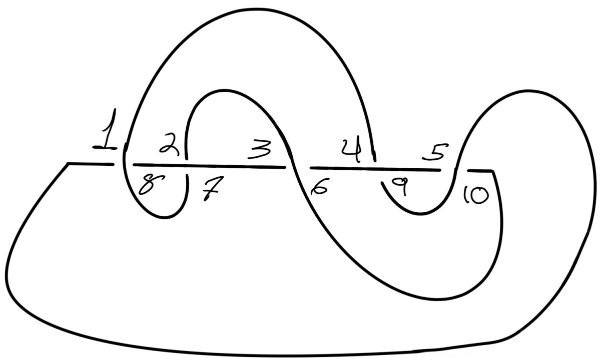
FIGURE 7. Trefoil

To encode that knot, what we did was to put numbers in all the crossings in order, following the knot. Then, we make a table and put in order all the odd numbers in the top and all the corresponding even number in the bottom. We get the following, which we call the *Dowker notation* of this knot.

1	3	5
4	6	2

Now, we will learn how to get a knot from a given Dowker's notation. We will use the information provided by the numbers and with this information we will graph it.

1	3	5	7	9
8	6	10	2	4



First, what we did to graph this knot is to make an horizontal line, and and we start by putting the crossings. The numbers on the top of the line should be in order (1,2,3,...) and these numbers should be connected with the others numbers that the Dowker's notation shows.

$$(1, 8)\ (2, 7)\ (3, 6)\ (4, 9)\ (5, 10)$$

Remember that you cannot repeat the same number in the knot, and we choose whether the crossings are alternating or not, so let us choose them to be alternating after the crossing $(5, 10)$. We have to connect the line that we made in the beginning with the crossing $(6, 3)$ because we cannot repeat a number, and after that point, we still continue doing the same until we complete the knot.

5. Knots and Sticks

What if we tell you that you can make knots with sticks? In the real world, making knots with sticks is a usual way to construct a knot. To start making a knot with sticks, you should now that minimum number of sticks that you need to make a knot is 3 sticks. This should give you an unknot. We will show and example.

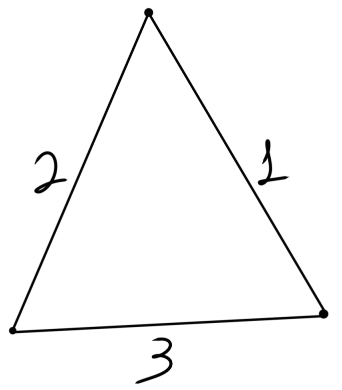


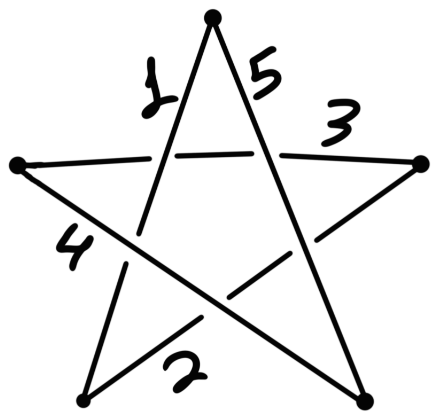
FIGURE 8. Unknot with sticks

Now, let us prove a theorem that we discovered with the help of our mentor. (We do not know if it is already known):

Theorem 5.1. *(EJ crossing theorem) With $n+3$ sticks, we can construct a projection of the unknot with n crossings .*

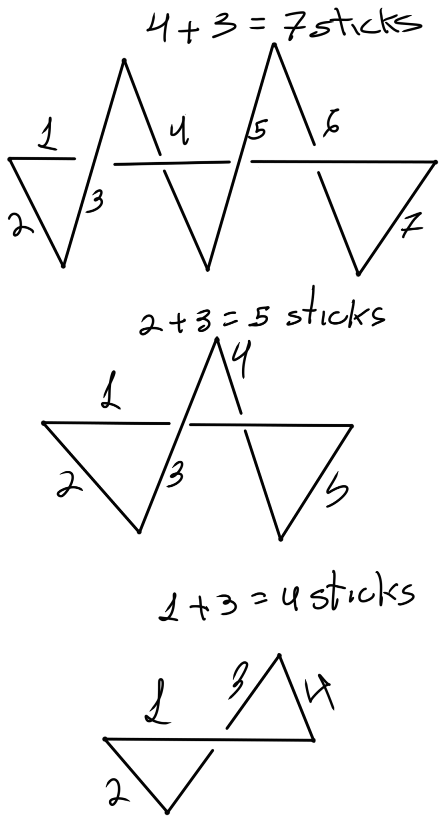
Proof of Theorem. Let us make the explicit construction. First, let us start with one stick. Now, attach another stick to one of the ends of the first stick. Now, we add a third stick to the end of this second stick in such a way that is crosses the original stick. We continue this process, in such a way that the k-th stick we add crosses the original stick a $(k-2)$-th time. Finally, after adding the $n+2$ stick, we will have our n crossings, so we close this knot with a $(n+3)$-th stick that is not going to cross the original stick. Also, by the way in which the crossings are alternated, we can verify that the resulting knot is a projection of the unknot. Hence, $n+3$ sticks are enough to create a knot with sticks with n crossings, as desired. □

Remark 5.2. This formula gives us a way to get a projection of the unknot with n crossings, but it is not always the one with the minimum number of sticks. For example, note that we can have 5 crossings with only 5 sticks.



Let us make some visual examples to help us understand the proof. In general, our construction will look like a zigzag line.

- If $n=4$, we want a knot with four crossing. Following the formula, we will replace the n by 4 and this would give us $4+3$, which would be equal to 7 sticks that we are going to use to create our knot.
- If $n=2$, we want two crossings, so we are going to do the same: $2+3=5$ sticks, and so we create our second knot joining the sticks from end to end.
- If $n=1$, we will want a one crossing, so we will follow the formula, so we will need $1+3=4$ sticks.



Now, let us talk about nontrivial knots with sticks. As any knot, a nontrivial knot can be made with sticks. However, it must have at least 3 crossings, as we mentioned before. Thus, checking the cases for 3, 4, and 5 sticks, we realize that the smallest number of stick we need to make a nontrivial knot with sticks is at least 6 sticks. For example, with 6 sticks we can construct a trefoil, as it is shown below.

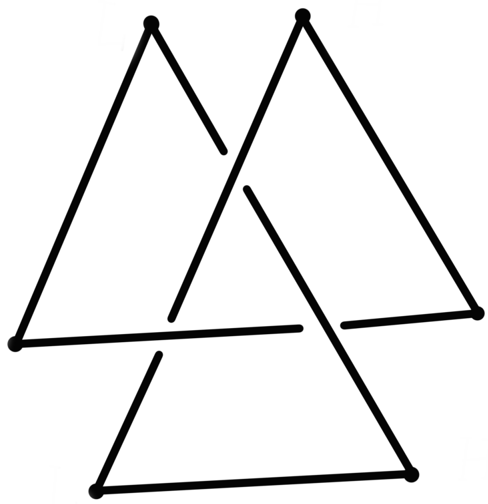
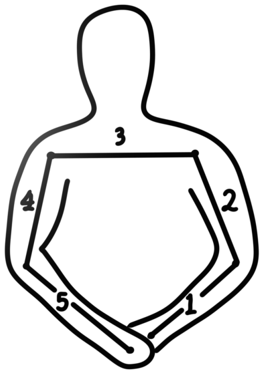
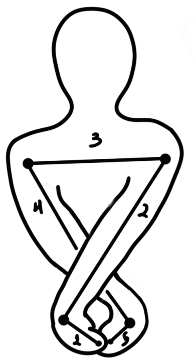


FIGURE 9. Trefoil with sticks

Is it possible to do a knot with our body? Indeed, it is. If you put your hands together, you will be a knot with five sticks. However, the real question is: can you make a crossing with your arms?



Well, it is complicated or impossible to do it with the "sticks" as in the picture because our composition does not allow us, but if we unite the sticks (1,2) and (4,5) and we also use our hands as sticks, it is possible.



References

[1] C. Adams. *The Knot Book*.

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