GEOMETRIC APPROACH TO INVESTIGATING PROPERTIES OF TARSKI GROUPS

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Abstract. We discuss Tarski Monster Groups –infinite groups G such that every non-trivial proper subgroup is of prime order. With many applications to combinatorics, geometry, and topology, these groups have many unusual properties that we will investigate. After developing a geometric approach to the construction of proofs, we will conclude with the proof of existence.

1. Introduction

Group Theory, as a branch of mathematics, is a chiefly the study of symmetry. In nature objects have many beautiful symmetries: radial symmetry, fractal symmetry, spiral symmetry, crystal symmetry, etc. This exists in mathematics as well and group theory is the algebraic language used to describe it. The groups that we will be dealing with in this paper are of a more abstract variety however, the basic intuition that comes from geometry is a helpful aid in understanding. This section serves as a prelude to the fundamental concepts and principles of group theory that will be useful in the discussion to come.

1.1. **Basics of group theory.** We will start with defining a group.

1.1.1. *Definitions.*

Definition 1. *A group is an ordered pair $(G, *)$ where G is a set and $*$ is an algebraic operation (binary operation) on G satisfying the following axioms:*

- $(a * b) * c = a * (b * c)$, *for all* $a, b, c \in G$.
- *there exists an element e in G called an identity of G, such that for all $a \in G$ we have $a * e = e * a = a$*
- *for each $a \in G$ thee is an element a^{-1} of G, called an inverse of a, such that $a * a^{-1} = a^{-1} * a = e$.*

If a group operation is commutative, we call the group an abelian group (Norweigan mathematician Niels Henrik Abel). We see that if e is identity for G then, $f(e)$ is the identity of H, that H is Abelian if, and only if, G is Abelian.

Definition 2. *A group G is called finite if G is a finite set. If not, we say G is an infinite group. The number of elements in a finite group is called its order. We write this as $|G| = n$ for n elements in set G.*

Definition 3. *A group isomorphism is a bijective mapping f of a group G onto a group H if for any $x, y \in G$ if f preserves the group operation. We write this as $G \cong H$.*

In other words, an isomorphism captures the notion that two groups have identical algebraic structures, despite having different elements and operations. To illustrate this, consider the symmetries of an equilateral triangle (S_3).

1.1.2. *Cyclic groups & Cosets.* A subgroup is a subset of a group that forms a group itself, retaining the same group operation. More formally,

Definition 4. *A subgroup is a group G is a non-empty subset $H \subset G$ such that,*

- *the product ab is in H when $a \in H$ and $b \in H$*
- *if $a \in H$ then $a^{-1} \in H$.*

There are two obvious examples of subgroups that every group, G, has a subgroup e consisting of only the identity, improper subgroup, and a subgroup containing all elements of G, proper subgroup.

Definition 5. *A cyclic group is a group that can be generated by a single element a. a is called the generator.*

In number theory, congruence modulo allow us to study numbers based on their remainders when divides by a given modulus. Cyclic groups provide a natural framework to explore these congruence's, in particular, to the set of integers modulo n ($\frac{Z}{nZ}$).

Theorem 1. *Every subgroup of a cyclic group is cyclic.*

Proof. Consider a subgroup H of a cyclic group. Since H is a subgroup, it must also satisfy the group axioms and contain the identity element of the larger group. This means that it contains at least one element. Now, let's find a generator for H – an element that can produce all other elements in H through repeated application of the group operation. Since, the larger group is cyclic we can express its elements as power of g. We know that H is closed so, we can examine the powers of g within H.Since H is a subgroup, if we take the power of any element in H, the result will also be in H. In other words, if we take g_n, where n is an integer, and g^n is in H, then the powers of g_n, such as $(g^n)^2$,

$(g^n)^3$, and so on, will also be in H. Notice that the powers of g^n within H cover all of the element cover H. ☐

Next, our focus turns to cosets. Cosets and relations play an important role in the factorization of groups and are often used to understanding subgroup structure.

Definition 6. *Any subgroup $H \subset H$ can be shifted to the left by an arbitrary element $a \in H$. The subset aH is called a left coset of $H \in G$. The same follows for the right coset.*

Definition 7. *A group homomorphism is a map $f : G \to H$ between two groups such that the group operation is preserved.*

A relation of two sets is a subset of the Cartesian product of their elements which consists of all possible ordered pairs. They describe the connection and association between elements of one or more sets. Defining a notion of equivalence in group theory, equivalence relations provide a way to classify elements into distinct equivalence classes based on certain shared properties or characteristics.

Definition 8. *An equivalence relation on a set G is a binary relation that satisfies three properties*

- *Reflexivity: every element in G is related to itself.*
- *Symmetry: If two elements x and y are related, then y is also related to x.*
- *Transitivity: if two elements x and y are related, and y and z are related, then x and z are also related.*

1.2. **Types of groups.** Groups frequently arise as permutations or symmetries of collections of objects. For instance, consider the rigid motions in R^2 that preserve a particular regular n-gon.This collection of actions is called a Dihedral group. Other examples includes:

1.2.1. *p-groups.* A p-group is a specific type of group characterized by its order and certain properties related to a prime number, denote p. In a p-group, the order of the group (the number of elements it contains) is a power of prime p. More formally, if G is a p-group, then the order of G is p^n for some non-negative integer n. . P-groups are often significant in the study of modular arithmetic, congruence, and the study of primes.

1.2.2. *Free Burnside problem.* The Free Burnside Problem, is a famous problem in group theory that originated from the work of British mathematician William Burnside in the early 20th century.

Definition 9. *A free group is one for which no relation exists between its group generators other than the relationship between an element and its inverse required as one of the defining properties of a group.*

Definition 10. *A free Burnside group is a free group in the variety of Abelian group. In other words, $B(A, n) = F/F^n$ where $F = F(A)$.*

A simple formulation of the Burnside problem is as follows: I. If G is a finitely generated group with exponent n, is G necessarily finite and II. for which positive integers m, n is the free Burnside group $B(m, n)$ finite?

No complete solution has yet been produced for this problem however, advances were made in the 1960s by Golod, Kostrikin, Novikov, and Adian. Famously it was said "this paper is possibly the most difficult paper to read that has ever been written on mathematics." I will not be presenting their arguments here. However, it is important to recognize the innovation of their approach, namely their geometrization. This method now becomes available to a wider range of problems in combinatorial group theory such as the one we will be discussing.

2. Geometric Approach

This approach to looking at relations in groups will be more clear if we first formally define a number of topological concepts:

2.1. **Two-dimensional topology.**

Definition 11. *A Family of sets is a collection F of subsets of a given set, S. This is also called a family of sets over S.*

Definition 12. *A topological space is an arbitrary set G with a distinguished family of subsets N such that*

- *$G \in N$, and the null set is in N*
- *the union of any family of sets in N is also in N, and*
- *the intersection of a finite family of sets in N is also in N.*

We say something is a metric space when a mathematical structure consists of a set together with a distance function that satisfies certain properties. This provides a framework for measuring distances between elements and studying convergence. We call some topological space a *quotient space* when it is formed from gives spaces X and Y by pasting them together and identifying two homomorphic subspaces. We can do something similar by introducing an equivalence relation and identifying equivalent points.

2.1.1. *The Jordan curve theorem.* Note that a closed curve in a topological space is a subspace homeomorophic to a circle, a Jordan arc in that space is a subspace homeomorphic to a closed interval, and the points of a subset which do not lie on its boundary are called it's interior points. Knowing this, the Jordan curve theorem states that

Theorem 2. *For any closed curve C in the Euclidean plane R^2, its complement consists of two connected components such that C is their common boundary. If C is an arc in the plane, then its complement is connected.*

This leads us finally to the combinatorial definition of a surface.

2.1.2. *Combinatorial Definition of a Surface.* Without reference to any specific geometric embedding or coordinate system, we define a surface by constructing it from a collection of polygonal regions and specifying how these regions are glued together.

By specifying the faces, their edges, and the "gluing instructions" a combinatorial surface can be completely described. The result is a combinatorial abstraction of a surface that captures its connectivity and local structure without explicitly considering its geometric properties.

2.2. **Van Kampen Diagrams.** In 1933, Van Kampen published a paper presenting a combinatorial tool to be used in algebraic topology to visualize and compute groups of topological spaces. He did this by representing them as unions of simpler spaces through decomposing a given space into a collection of simpler subspaces. In the next section we will show how this is use in the geometrical deduction of consequences of relations in groups.

Definition 13. *A word refers to a sequence of elements from a given group.*

2.2.1. *Example.* If the relations $a^3 = 1$ and $bab^{-1} = c$ hold in some group, then it obviously follows that $c^3 = 1$. This deduction can be represented as a circuit a triangular cell yielding the "word" a^3 and a circuit of each of the 4-sided cells yielding $cba^{-1}b^{-1}$. On the boundary of the whole figure we read c^3, which is the left-hand side of the given consequence of the relations $a^3 = 1$ and $cba^{-1}b^{-1}$. We can now formalize this with the following definition:

Definition 14. *A diagram on a surface M over a presentation is any diagram α over the alphabet β whose cels are all γ-cells or 0-cells for some γ.*

3. Proof for the Existence of Tarski Groups

To understand the idea behind Olshanskii's approach, let us first consider the concept of a Tarski group. A Tarski group is a finitely generated group with the property that it is isomorphic to its proper subgroup. The key idea of this proof is to find a certain pattern within a diagram that allows the construction of an infinite sequence of words, each representing an element in the group. These words generate a subgroup that is isomorphic to the original group. Thus The Tarski problem is formulated as

3.1. **Tarski's Problem.** Do there exist infinite groups all proper subgroups of which are of fixed prime order p?

Definition 15. *Quasi-finite groups are infinite groups where all proper(of prime order) subgroups are finite.*

3.2. **Construction of Non-Abelian quasi-finite groups.** Let $G(\infty)$ be a group with the relations defined by a set of conditions R that must be satisfied. For the sake of simplicity, we will not go into all of these and their justifications but the full proof may be referenced. This relation is of the form $T_1A^nT_2\cdots T_hA^n = 1$.

Theorem 3. *If* $|A| > 1$*, then* $G(\infty)$ *is infinite.*

Theorem 4. *The centralizer of a non-trivial element* $X \in G(\infty)$ *is cyclic. Every abelian or finite subgroup* H *of* $G(\infty)$ *is cyclic.*

Definition 16. *A relator is a type of relation that involves a word or an equation formed by generators.*

For example, consider a group presentation with generators a,b, and c and the following relator: $ab^2c = 1$. This relator imposes a constraint on how these generators can be combined. We define an alphabet A composed of a_1 and a_2 and start by imposing restriction on it. We introduce the following relation

$$A^n = 1.$$

For each period $A \in x$ we fix a maximal subset y such that:

- if $T \in y$, the $1 \le |T| < d|A|$
- each double coset of subgroups of $G(i)$ contains at most one word in y with a minimal length representing this double coset.

It then follows from its definition, that every word in y is minimal in rank $i-1$ and is not equal in rank $i-1$ to a power of A. We now proceed to verify each of the conditions for R by induction. Again, for the sake of simplicity and with the aim of conveying the overall approach we will not cover each step here. This is an example of a constructive proof which establishes the existence of an object by explicitly constructing the desired object.

Lemma 1. *The presentations of the groups $G(i)$ satisfy condition R.*

We can now generate pairs for $G(\infty)$.

Lemma 2. *All proper subgroups of $G(\infty)$ are abelian.*

Proof. Let H be a non-abelian subgroup. We may assume that H contains elements F and that satisfy $|T| < 3|F| < d|F|$, if follows from the definition of the relations that $a_1, a_2 \in < F, T > \subset H$. So, $H = G(\infty)$. □

Theorem 5. $G(\infty)$ *is an infinite group all of whose proper subgroups are cyclic of prime order p.*

Proof. By Lemma 1 the infiniteness of $G(\infty)$ follows from Theorem 4. Furthermore, by Lemma 2, it is sufficient to consider only abelian subgroups of $G(\infty)$. By Theorem 3, such a subgroup is of necessity cyclic. A generator X satisfies the equation $X^p = 1$ by Theorem 3, since in the definition of $G(\infty)$ all n_A are equal to $n_0 = p$. By this construction, Tarski groups exist. □