Introductory Combinatorics and Its Applications

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1 Introduction

Combinatorics is centered around the most fundamental concept of mathematics: counting. This paper will explore basic enumerative combinatorics, including permutations, strings, and subsets and how they build on each other. Later, we will explore applications of these concepts in subjects such as Ferrrers shape, the binomial theorem, and Pascal's triangle. Finally, we connect these topics to their applications in other branches of combinatorics and mathematics.

2 Introductory Counting Problems and Partitions

We will begin this paper by introducing permutations, which help us order and arrange a set of variables in a fairly intuitive manner.

Theorem 2.1. *The number of permutations of an n-element set is $n!$.*

Proof. We operate by the convention that $0! \equiv 1$. If we assume n people arrive at a dentist's office and the dentist treats them one by one, how many different orders that each patient will be served are possible? There are n possible choices for the person who is served first, and, consequently, $n-1$ for the patient who will be seen second. Therefore, the number of orders in which the patients can be treated is $n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$.

Definition 2.1. The arrangement of different objects into a linear order using each object exactly once is called a *permutation* of these objects. The number $n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ of all permutations of n objects is called n *factorial*, and is denoted by $n!$.

□

Problem 2.1. *If 8 people enter a dentist's office the dentist treats them one by one, in how many different orders can the patients be served. There are* $8! = 40320$ *different orders that each patient will be served.*

Solution. *There are* $8! = 40320$ *different orders that each patient will be served.*

Now, previously we were operating on the assumption that all of the objects within our set are different. If there are, however, objects that are identical within our set, one labels this a *multiset*. A linear order that contains all the elements of a multiset exactly once is called a *multiset permutation*.

Theorem 2.2. *Let* n*,*k*,*a_1*,*a_2*, ... ,* a_k *be non negative integers satisfying* $a_1 + a_2 + \cdots a_k \equiv n$*. Consider a multiset of* n *objects, in which* a_i *objects are type* i*, for all* $i \in [k]$*. Then the number of ways to linearly order these objects is*

$$\frac{n!}{(a_1! \cdot a_2! \cdot \cdots \cdot a_k!)}$$

Proof. A gardener has five red flowers, three yellow flowers, and two white flowers to plant in a row. In how many different ways can she do that? In other words, how many permutations does our multiset have? We can solve this by beginning by assuming that all objects in our set are different. The gardener differentiates her flowers by sticking different labels on each one. We are left with ten different flowers, meaning we can arrange them in 10! different ways. Now, we must identify which flowers differ only because of their labels. The five red flowers can be given five different labels in 5! different ways, the yellow 3!, and the white 2!. Moreover, the arranging of flowers of different colors is done independently of each other. This means that the labeling of all ten flowers can be done in $5! \cdot 3! \cdot 2! \cdot$ different ways once the flowers are planted in any of A different ways. Therefore $A \cdot 5! \cdot 3! \cdot 2! \equiv 10!$, or, in other words,

$$A = \frac{10!}{5! \cdot 3! \cdot 2!} = 2520$$

□

Problem 2.2. *A traveling agent has to visit four cities, each of them five times. In how many different ways can he do this if he is not allowed to start and finish in the same city?*

Solution. *There are* $\frac{20!}{5!\cdot5!\cdot5!\cdot5!}$ *ways to visit four cities, each of them five times. The number of ways to do this so that we start and end in the same city, city A, is* $\frac{18!}{5!\cdot5!\cdot5!\cdot3!}$*, as we are free to choose the order in which we make the remaining 18 visits. The* $3!$ *comes from the fact that 3 of the cities we get to choose to visit are city A. The same argument applies for the number of visiting arrangements that start and end in B, that start and end in C, and that start and end in D. So the final answer is*

$$\frac{20!}{5!\cdot5!\cdot5!\cdot5!} - 4\cdot\frac{18!}{5!\cdot5!\cdot5!\cdot3!}.$$

Next, we will examine problems that construct strings from a finite set of symbols, referred to as a *finite alphabet*, which only requires that each symbol appears at most once, but not that each symbol occurs a certain number of times, as opposed to permutations.

Theorem 2.3. *The number of k-digit stings over an n-element alphabet is* n^k

Proof. The first digit can be chosen in n different ways; then, the second digit can also be chosen in n different ways because we are allowed to use the same digit again. If we follow this pattern then, we can choose the kth element in n different ways. As we have made each of these choices independently from one another, the total number of choices is n^k ☐

The number of k-digit positive integers is $9\cdot10^{k-1}$

Solution. *We have 9 choices for the first digit (everything but 0), and ten choices for each of the remaining* $k-1$ *digits. Therefore, the number of total choices is* $9\cdot10\cdot10\cdots10 = 9\cdot10^{k-1}$.

Theorem 2.4. *Let* n *and* k *be positive integers satisfying* $n >= k$*. then the number of k-digit strings over an n-element alphabet in which no letter is used more than once is*

$$n(n-1)\cdots(n-k+1) \equiv \frac{n!}{(n-k)!}$$

Proof. Similarly to factorials, we have n choices for the first digit, $n-1$ choices for the second digit, and so on. The distinction is that we do not necessarily use all our n objects, we stop after choosing k of them. ☐

Suppose we want to choose a subset of k students from a total of n to form a school committee. We have n choices for the first student, $n-1$ choices for the second student, and $n-k+1$ choices for the kth student, for a total of $\frac{n!}{(n-k)!}$. However, within this subset, the order of the students does not matter, so we must divide by the number of permutations of our subset, or $k!$. Thus, the total number of *combinations* we have is

$$\frac{n!}{(n-k)!k!}.$$

Definition 2.2. Given a set of $n!$ objects, the number of combinations of k objects where *order doesn't matter* is given as

$$\frac{n!}{(n-k)!k!} = \binom{n}{k}.$$

This is read as "n choose k", and is also known as a binomial coefficient.

Now we will delve into integer partitions. Our goal is to find the positive n as a sum of positive integers. Here, the order and the summands do not matter. For example, $3 = 2 + 1$ and $3 = 1 + 2$ count as only one way of writing 3 as a sum of positive integers.

Definition 2.3. Let $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ be integers so that $a_1 + a_2 + \cdots + a_k$=n. Then sequence $(a_1, a_2, \cdots, a_k)$ is called a *partition* of the integer n. The number of all partitions is denoted by $p(n)$. The number of partitions of n into exactly k parts is denoted by $p_k(n)$.

We want to mention that the approximate size of $p(n)$ is provided by the following formula, though it is much too complex to prove in this paper. This formula shows us that $p(n)$ grows faster than any polynomial, but slower than any exponential function.

$$p(n) \approx \frac{1}{4\sqrt{3}} exp(\pi\sqrt{\frac{2n}{3}})$$

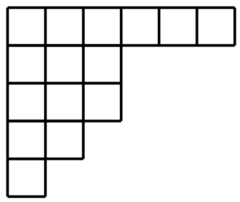
We will now introduce an interesting identity of partitions. The main objective of proving these identities is the following graphical representation of partitions.

2.1 Ferrers Shape

A **Ferrers shapes** of a partition is a set on n square boxes with horizontal and vertical sides so that in the ith row we have $x_i$boxes and all rows start with a vertical line. So, n_1 will correspond to the first row; n_2 will correspond to the second, and so on.

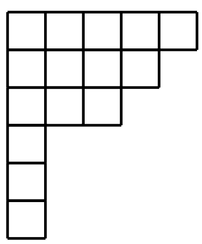
Problem 2.3. *Let P be the partition of 15; Ferrers shape for P is:*

$$P = 6 + 3 + 3 + 2 + 1$$



The reflection of Ferrers shape of a partition is where we can observe an interesting transformation. Upon it's reflection with respect to it's diagonal, another shape is formed representing the *conjugate* partition. This way the ith row of the conjugate partition of p is as long as the $i$the column of the Ferrers shape of p. Another way to think about it is, initially to obtain the partition of n from Ferrers shape, you would read horizontally to find the corresponding parts. Now, we will read vertically to find it's conjugate.

Problem 2.4. *Let P be the partition of 15; Ferrers shape for P is:*

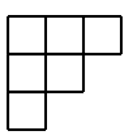


$$P = 5 + 4 + +3 + 1 + 1 + 1$$

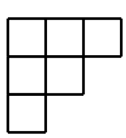
Definition 2.4. A partition of n is called a *self-conjugate* if it is equal to its conjugate.

Let P be the partition of 6; Ferrers shape for P is:

$$P = 3 + 2 + 1$$



The conjugate of P of 6 is:

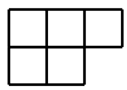


$$P = 3 + 2 + 1$$

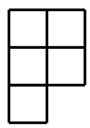
So, the partition of 6 is a self-conjugate

Let P be the partition of 5; Ferrers shape for P is:

$$P = 3 + 2$$



The conjugate of P of 5 is:



$$P = 2 + 2 + 1$$

So, the partition of 5 is not a self-conjugate

Ferrers shape can now aid us in proving many theorems relating to partitions.

Theorem 2.5. *The number of partitions into at most k parts is equal to that of partition of n into parts not larger than k.*

Proof. The first number is equal to that of Ferrers shape of size n with at most k rows. The second number is equal to that of Ferrers shapes with at most k columns. Finally, these two sets of Ferrers shapes are equinumerous as one can see by taking conjugates. □

Problem 2.5.

The partition of 8 into 3-parts is:

$$(6 + 1 + 1)$$
$$(5 + 2 + 1)$$
$$(4 + 3 + 1)$$
$$(4 + 2 + 2)$$
$$(3 + 3 + 2)$$

The partition of 8 into parts the largest size of which is 3 is:

$$(3 + 1 + 1 + 1 + 1 + 1)$$
$$(3 + 2 + 1 + 1 + 1)$$
$$(3 + 2 + 2 + 1)$$
$$(3 + 3 + 1 + 1)$$
$$(3 + 3 + 2)$$

As we can observe, none of the values in the second column go beyond 3 and there are 5 partitions on each side.

3 The Binomial Theorem and Combinatorial Identities

The binomial theorem and identities on binomial coefficients are a crucial part of combinatorics not only for what they tell us about these subjects, but also the method that we use to prove them. To prove these statements, we will show that both sides of the equation are counting the same kinds of objects.

3.1 The Binomial Theorem

Suppose you have the sum $(x + y)$, raised to a power of 2. Expanding this is simple:

$$(x + y)^2 = x^2 + y^2 + 2xy.$$

However, suppose now that the sum $(x + y)$ is raised to the power of a non-negative integer n. If n is a large integer, expanding this equation by hand is tedious and very prone to error. Enter, the binomial theorem.

Theorem 3.1. *For all nonnegative integers n,*

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. To expand this sum, we must choose one summand from $(x + y)$ and multiply it with another in a different set of parenthesis. Since there are 2 terms to choose from in each set of parenthesis, and there n sets of parenthesis, the total possible number of ways to do this is 2^n. So, on the left side of the equation, we will have the sum of 2^n products. On the right hand side of the equation, the sum

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

counts the number of subsets of an n-element set by counting the subsets of size $0, 1, \ldots, n$ and adding them together. This may also be done by Theorem 2.3, where our alphabet consists of two possible options: an element is either in the set, or it is not in the set, for a total of 2^n combinations. Now, we see that the number of elements on both sides of the equation are equal. Next, in order to form each term of $x^k y^{n-k}$, as will be on the left hand side when we expand, we must choose a set of k terms to contribute a factor of x, and the remaining will contribute a factor of y. There are $\binom{n}{k}$ k-element subsets of the set of n summands, so we will get each term $\binom{n}{k}$ times. Since this is shown on the right hand side of the equation, our proof is complete. □

Note that another proof of the binomial theorem can be achieved by using mathematical induction. However, for the purposes of this section, a combinatorial proof is presented to highlight the method of showing both sides of a given equation count the same kinds of objects.

3.2 Identities of Binomial Coefficients

Here, we will continue our study of binomial coefficients as introduced in Section 2. We will also see a mix of combinatorial and computational proofs, as well as several that use the binomial theorem.

Theorem 3.2. *For all nonnegative integers n,*

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

Proof. To begin our proof, we claim that both sides of the equation count the number of subsets of a set of n objects. The left hand side creates a bijection over a 2-element alphabet (the element is in the subset or it is not), and the right hand side calculates all k-element subsets of the set of n elements and adds them together. □

Theorem 3.3. *For all positive integers n,*

$$\sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} = 0.$$

Proof. Here, we want to prove that the alternating sum of the binomial coefficients $\binom{n}{k}$ is equal to zero. This is a direct application of Theorem 3.1 if we let $x = -1$ and $y = 1$. By the binomial theorem,

$$\begin{aligned}(-1+1) &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cdot 1^{n-k} \\ &= \sum_{k=0}^{n} \cdot (-1)^k \cdot \binom{n}{k}\end{aligned}$$

and our proof is complete. □

We will prove the following theorem by using the definitions we have covered and cancellation techniques.

Theorem 3.4. *For all nonnegative integers k and n, such that $k \leq \frac{n-1}{2}$, the inequality*

$$\binom{n}{k} \leq \binom{n}{k+1}$$

holds.

Proof. To begin, we would like to obtain our condition $k \leq \frac{n-1}{2}$ from the equation given. From definition 2.2, we may rewrite the equation as

$$\frac{n!}{k!(n-k)!} \leq \frac{n!}{(k+1)!(n-k-1)!}.$$

Multiplying by $k!$ and $(n-k-1)!$, as well as dividing by $n!$ gives us

$$\frac{1}{n-k} \leq \frac{1}{k+1},$$

which may then be rearranged to

$$k \leq \frac{n-1}{2}.$$

This completes our proof. $\square$

3.3 Pascal's Triangle

To construct Pascal's triangle, each element in a row will be the sum of two adjacent elements in the preceding row, starting with an infinite row of zeros, with an exception of a single 1 in row $n = 0$. The first seven rows of the triangle are shown below. Many beautiful mathematical properties and patterns can be found in this diagram.

n=0							1						
n=1						1		1					
n=2					1		2		1				
n=3				1		3		3		1			
n=4			1		4		6		4		1		
n=5		1		5		10		10		5		1	
n=6	1		6		15		20		15		6		1

Notice that, in each row, we have actually seen these numbers before! The numbers in the nth row of Pascal's triangle correspond to the binomial coefficients of the expansion of $(x+y)^n$.

In addition, Theorem 3.2 can be seen in the sum of the elements of each row of the triangle; row zero adds to $2^0 = 1$, row one adds to $1 + 1 = 2^1$, row two adds to 2^2, and so on.

Finally, notice that if we start with the rightmost element of the kth row and descend diagonally (down and to the left), the sum of all numbers in this progression is also an entry of the triangle: as an example, starting in the leftmost element of the third row (1) and descending diagonally to 3 and 6, our sum is the binomial coefficient a row down and to the right (10).

This demonstrates another interesting property of binomial coefficients:

Theorem 3.5. *For all nonnegative integers k and n,*

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

4 Conclusion

As seen in previous sections, combinatorics uses seemingly simple concepts to evaluate and solve complex problems. A solid understanding of basic concepts like permutations, alphabets and strings, and binomial coefficients is crucial to understanding more complex topics such as Ferrers shapes and combinatorial proofs. Not only applicable to these areas of mathematics, a basic understanding of combinatorics is beneficial in theoretical computer science, graph theory, and group theory. For instance, combinatorial algorithms are useful in asking how efficiently a said task can be completed, which has applications in computer science sorting algorithms. Overall, studying combinatorics equips you with tools needed to solve more complex problems in various areas of math, and makes the connections between these areas of study much clearer.

Acknowledgements and Resources

References

[1] M. Bona *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory*, 2017.

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