

## The PRIMES 2011 problem set

Dear PRIMES applicant,

This is the PRIMES 2011 problem set. Please choose at most **eight** of these problems and send us their solutions at [primes@math.mit.edu](mailto:primes@math.mit.edu) by December 15, 2010.

You can type the solutions or write them by hand and then scan them; please save your work as a DOC, PDF, or JPG file.

Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.

You are allowed to use any resources to solve these problems, *except other people's help*. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

Note that many of these problems are tricky, and we do not expect many applicants to be able to solve eight of them. We recommend that you think about them, on and off, over some time (several days), and encourage you to apply if you can solve **at least three** of the problems (or solve a number of parts of various problems which is roughly equivalent). There will be many factors in the admission decision besides your solutions of these problems.

Although any applicant is allowed to choose any problem, we note that problems 1, 5, 7, 8 are especially relevant for computational biology applicants, while problems 4, 9, 11, 12 are especially relevant for mathematics applicants. Also, Problems 10, 11, 12 are more difficult than others. Problem 10 requires some familiarity with linear recursions, Problem 11 with linear algebra, and Problem 12 with analysis. If you are not comfortable with these topics, you may skip some or all parts of these problems, and should not feel discouraged.

Enjoy!

**Problem 1.** You roll four dice until you have at least two equal numbers (i.e., if all the four numbers are different, you just roll the dice again). Then, you win if at least one of the four dice shows a 3. What's your chance of winning? (Represent the answer as a fraction).

**Problem 2.** Let  $125! = 5^k N$ , where  $k$  is a positive integer, and  $N$  is a positive integer not divisible by 5 (such  $k$  and  $N$  are unique).

(a) What is  $k$ ?

(b) What is the remainder of division of  $N$  by 5?

**Problem 3.** How many integers between 1 and 1,000,000 cannot be written as a sum of two or more consecutive positive integers? What are these numbers?

**Problem 4.** What are the first 100 digits after the decimal point of the number  $(\sqrt{10} + 3)^{2010}$ ? (You may compute a few powers of  $\sqrt{10} + 3$  by calculator, observe a pattern, then try to prove).

**Problem 5.**

(a) Mary lives on the corner of 7th avenue and 54th street in Manhattan. Every night she takes a 10-blocks-long walk which begins and ends at her home. How many different walks can she take?

(b) How many different walks can Mary take if her walks are  $2n$  blocks long? (we assume that Manhattan is an infinite rectangular lattice).

(c) Suppose Mary is an absent-minded mathematician, and takes a 20-blocks-long walk which starts at her home but does not necessarily end there. What's the chance she actually does return home at the end of the walk, given that at each corner she picks the direction randomly with equal probabilities (i.e., each direction with probability  $1/4$ )? Compute the approximate value, using Stirling's formula  $n! \cong (n/e)^n \sqrt{2\pi n}$ .

**Problem 6.** The vertices of a triangle  $T$  in the plane have rational coordinates. Can its sides be in proportion  $2 : 3 : 4$ ?

Hint: Use Heron's formula for the area of a triangle.

**Problem 7.** In an hour, a bacteria dies with probability  $p$  or else splits into two. What is the probability that a single bacteria produces a population that will never become extinct?

Hint: Let  $p_n$  be the probability of the population to die out by the  $n$ -th generation. Write a recursive equation for  $p_n$ , compute its limit, and express the answer via this limit.

**Problem 8.** Consider a population of unicorns. Unicorns are diploids, i.e., they have two copies of each gene, say  $A$  and  $B$ , which may be normal or mutated (i.e., defective). When a baby unicorn is born, each of the two parents passes one of these two copies to the baby.

If only *one* of the two copies is mutated, then the baby unicorn is a *carrier* of a genetic disease; i.e., it does not actually have the disease but may pass it to descendants. However, if *both* of these copies are mutated, then the baby unicorn is born sick and will not survive to adulthood (thus not producing descendants).

(a) If a unicorn carries a mutated copy of a certain gene, then for each of its siblings, what is the probability of having a mutated copy of the same gene? (You may neglect the rare possibility that both parents carry a mutation in this gene).

The genome of a unicorn has 8000 genes. Also, it is known that for any given adult unicorn and any given gene, with probability  $1/1000$  one of the copies of this gene ( $A$  or  $B$ ) in this unicorn is mutated, and these mutations are statistically independent for different genes.

(b) What is the probability that the baby unicorn has a genetic disease if its parents are siblings?

(c) if its parents are unrelated?

**Problem 9.** Let  $n$  be a positive integer. Find all the odd numbers between 1 and  $2^n$  which can be remainders modulo  $2^n$  of a perfect square.

Hint. Compute for the first few values of  $n$  to determine the pattern; then use induction.

**Problem 10.** (a) Let  $a_n(k)$  be the number of sequences of zeros and ones of length  $n$  which don't contain  $k$  zeros in a row. Find the recursive equation for  $a_n(k)$ , when  $k$  is fixed.

(b) If you toss a coin 10 times, what's the chance of not getting 3 heads in a row?

(c) Let  $p_n(k) = a_n(k)/2^n$  be the probability **not** to get  $k$  heads in a row from  $n$  coin tosses. Show that for each  $k$  there are constants  $C_k > 0$  and  $0 < b_k < 1$  such that  $p_n(k)/C_k b_k^n \rightarrow 1$  as  $n \rightarrow \infty$ . What is  $b_k$ ?

(d) Calculate  $a_n(2)$  and  $b_2$  explicitly.

**Problem 11.**

Let  $M(n)$  denote the space of all real  $n \times n$  matrices. Thus  $M(n)$  is a real vector space of dimension  $n^2$ . Let  $f(n)$  denote the maximum dimension of a subspace  $V$  of  $M(n)$  such that every nonzero element of  $V$  is invertible.

(a) Show that if  $n$  is odd, then  $f(n) = 1$ .

(b) Show that  $f(rs) \geq f(r)$  for any positive integers  $r, s$ .

(c) Show that for any  $n$ ,  $f(n) \leq n$ .

(d) Find  $f(2)$ .

(e) (harder) Find  $f(4)$ .

(f) (very hard) Find  $f(8)$ .

NOTE. There is a general formula for  $f(n)$ , but its proof is extremely difficult.

(g) What is  $f(n)$  if instead of real matrices we consider complex matrices? Rational matrices?

**Problem 12.** Let  $m$  be a positive integer, and let a sequence of positive numbers  $\{x_n\}$  be defined recursively by  $x_1 = 1/2$ ,  $x_{n+1} = x_n - x_n^{m+1}$ .

(a) Show that  $x_n \rightarrow 0$  as  $n \rightarrow \infty$ .

(b) Find  $a := \lim_{n \rightarrow \infty} \frac{\log x_n}{\log(1/n)}$ .

(c) Find  $K := \lim_{n \rightarrow \infty} n^a x_n$ .

Hint: for (b),(c), compare the sequence  $x_n$  to  $y_n = Cn^{-b}$ ,  $C > 0$ ,  $b > 0$ . Namely, express  $y_{n+1}$  as a function of  $y_n$ , and choose  $C, b$  so that this function is “as close as possible” to  $f(x) = x - x^{m+1}$ .