DETERMINANT FORMULAS FOR THE REFLECTION EQUATION ALGEBRA

DAVID JORDAN AND MASAHIRO NAMIKI

1. Introduction

The purpose of this note is to briefly report on the status of PRIMES project "Determinant formulas for quantum GL(N)", undertaken by PRIMES student Masahiro Namiki, and PRIMES mentor David Jordan. We are presently pursuing the project in collaboration with Samson Black, of Linfield College. After a brief introduction, we state conjectural results, and report on the status of those conjectures.

2. Braiding and R-matrix

Let $V = \mathbb{C}^N$, and let $\sigma : V \otimes V \to V \otimes V$ be a *braiding*, i.e. a linear isomorphism satisfying the following equality in $End(V \otimes V \otimes V)$:

$$(\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma) \circ (\sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma) \circ (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma). \tag{1}$$

Equation (1) is called the braid relation, and it allows one to define a representation of the braid group $B_N(\mathbb{C})$ on N-fold tensor products $V^{\otimes N}$ using σ.

Let $\mathrm{fl} : V \otimes V \to V \otimes V$ denote the flip map, $\mathrm{fl}(v \otimes w) = w \otimes v$. Let $v_1, \dots v_N$ be a basis for V. We collect the coefficients of $\mathrm{fl} \circ \sigma$ into an $N^2 \times N^2$ matrix R; thus, the entries of R are defined by the equation:

$$\sigma(v_k \otimes v_l) = \sum_{i,j=1}^{N} R_{kl}^{ij} v_j \otimes v_i. \tag{2}$$

Our primary example of a braiding is as follows. Let $q \in \mathbb{C}^\times$ and define σ by:

$$\sigma(v_i \otimes v_j) = q^{\delta_{ij}} v_j \otimes v_i + (q - q^{-1})\theta_{ji} v_i \otimes v_j, \tag{3}$$

where $\delta_{ij} = 1$ if $i = j$, 0 else, and $\theta_{ij} = 1$ if $i > j$, 0 else. Comparing Equations (2) and (3), we have:

$$R_{kl}^{ij} = q^{\delta_{ij}} \delta_{ik} \delta_{jl} + (q - q^{-1})\theta_{ij} \delta_{il} \delta_{jk}.$$

For example, for $N = 2$, relative to the basis $v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2$, we may write σ and R as matrices:

$$\sigma = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \qquad R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

3. Reflection equation algebra

Associated to the data (V,σ) is the *reflection equation algebra (REA)* $\mathcal{A}$. This algebra has N^2 generators a^i_j, for $i,j=1,\dots,N$, and relations:

$$\sum_{k,l,m,n=1}^{N} R^{ij}_{kl} a^l_m R^{mk}_{on} a^n_p = \sum_{s,t,u,v=1}^{N} a^i_s R^{sj}_{tu} a^u_v R^{vt}_{op}, \tag{4}$$

for $i,j,o,p=1,\dots,n$. Equation (4) is called the *reflection equation*, and it allows one to define a representation of the punctured braid group of $B_N(\mathbb{C}^\times)$ on $V^{\otimes N}\otimes M$, on certain $\mathcal{A}$-modules M satisfying a non-degeneracy property. For background on the REA, see [DM], [GPS].

Remark 3.1. It should be noted that the REA may be obtained from the so-called Faddeev-Reshetikhin-Takhtajan (FRT) algebra [FRT] via the "twist" procedure [DM], as follows. Let $\mathcal{C}$ denote the category of representations of the quantum universal enveloping algebra $U_q(\mathfrak{gl}_N)$, and let $\mathcal{C}^{op}$ denote the same abelian category with opposite tensor product $X\otimes^{op} Y := Y\otimes X$. Then the FRT algebra is an algebra in the tensor category $\mathcal{C}\boxtimes\mathcal{C}^{op}$, while the REA is an algebra in $\mathcal{C}$. The functor:

$$\mathcal{C}\boxtimes\mathcal{C}^{op} \xrightarrow{\mathrm{id}\boxtimes F_\sigma} \mathcal{C}\boxtimes\mathcal{C} \xrightarrow{\otimes} \mathcal{C},$$

where $F_\sigma:\mathcal{C}\to\mathcal{C}^{op}$ denotes the identity functor, equipped with a tensor structure σ, induces a functor from $\mathcal{C}\boxtimes\mathcal{C}^{op}$-algebras to $\mathcal{C}$-algebras, under which the REA is the image of the FRT algebra.

4. The center of REA

The center of the FRT algebra has been thoroughly studied and described: it is isomorphic to the sub-algebra of GL_N invariant matrix polynomials, which are the coefficients $\chi_1,\dots\chi_N$ of the characteristic polynomial:

$$\chi(A) = t^N - \chi_1 t^{N-1} + \dots \pm \chi_N$$

The center of the FRT algebra is thus generated elements $p_1,\dots,p_N$, which have formulas in terms of the generators a^i_j, which are quantum analogs of those for the χ_i. For example, we have:

$$p_1 = \sum_i q^{-2i} a^i_i, \qquad p_N = \sum_{\tau\in S_n} (-q)^{l(\tau)} a^1_{\tau(1)}\cdots a^N_{\tau(N)}.$$

The twist procedure outlined in the previous section induces an isomorphism of centers, so that the center of the REA of rank N is generated also by N elements, $\tilde{p}_1,\dots,\tilde{p}_N$, which are (different, ad-invariant) quantum analogs of the above coefficients. In particular, $\tilde{p}_1$ will be a quantum analog of the trace, while $\tilde{p}_N$ will be a quantum analog of the determinant. The goal of this research is to find explicit formulas for $\tilde{p}_1,\dots,\tilde{p}_N$ in terms of the generators a^i_j of the REA.

To begin, it is straightforward to see that:

$$\tilde{p}_1 := \sum q^{-2i} a^i_i$$

is a central element, deforming the trace function on matrices. Note that this is the same expression as for p_1 in the FRT algebra, because the twist procedure is identical on linear expressions in the generators. Computer experiments in MAGMA have motivated the following conjecture:

Conjecture 4.1. *The element,*

$$\tilde{p}_N := \sum_{\tau\in S_N} q^{e(\tau)}(-q)^{l(\tau)} a^1_{\tau(1)}\dots a^N_{\tau(N)},$$

is central in the REA, where $l(\tau)$ denotes the length *of a permutation, and $e(\tau)$ denotes the* excedance, *$e(\tau)=\#\{i \mid \tau(i)>i\}$.*

Example 4.2. For $N = 2$, we have:

$$p_N = a^1_1 a^2_2 - q^2 a^1_2 a^2_1,$$

while for $N = 3$, we have:

$$p_N = a^1_1 a^2_2 a^3_3 - q^2 a^1_2 a^2_1 a^3_3 - q^2 a^1_1 a^2_3 a^3_2 + q^4 a^1_2 a^2_3 a^3_1 + q^3 a^1_3 a^2_1 a^3_2 - q^4 a^1_3 a^2_2 a^3_1.$$

This conjecture has been checked for $N = 2, \ldots, 12$ by direct computation in MAGMA. It is interesting, firstly that such a simple deformation is possible (varying only the coefficients w.r.t the standard PBW basis), and secondly that the excedance statistic appears here. We note that it is essential that the covariant indices are permuted, while the contravariant indices are in standard order. It can be shown, already for $N = 3$, that such a simple formula for $\tilde{p}_N$ is not possible if the contravariant indices are permuted instead.

Analogously to the classical algebra of matrix coefficients, we hope to build the other central generators $\tilde{p}_i$ by combining the quantum trace and determinant formulas, analogously to the classical. For instance, let

$$det_q(i,j) := a^i_i a^j_j - q^{2(j-i)} a^i_j a^j_i.$$

Then we have the following

Conjecture 4.3. *For all N, the element:*

$$p_2 = \sum_{i<j} det_q(i,j) q^{-2(i+j)}$$

is central.

This conjecture was checked by direct computation for $N = 3, 4$. One end goal is to obtain explicit and combinatorially interesting expressions for the all central generators p_i.

We expect to prove Conjecture 4.1 in the near future, and are working on a framework for constructing the central generators $\tilde{p}_i$, along the lines of Conjecture 4.3.

References

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