Fibonacci Numbers and Continued Fractions

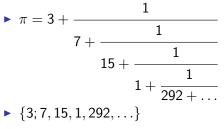
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Continued Fractions



▶ Continued fractions are very useful in approximation theory.
 ▶ π ≈ ²²/₇

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Continued Fractions

- $\{2; 1, 2, 1, 2, 1, \ldots\} = \{2; \overline{1, 2}\}$
- ▶ Rational ⇔ Finite Continued Fraction
- ► Irrational ⇐⇒ Infinite Continued Fraction
- Quadratic Irrational $(a + b\sqrt{c}) \iff$ Periodic Infinite Continued Fraction

Convergent: truncation of a continued fraction

Fibonacci Numbers

$$F_{n+2} = F_{n+1} + F_n, F_0 = 0, F_1 = 1$$

$$F_{n+1} = \{ \overbrace{1;1,1,1,1,\dots}^{n \ 1's} \}$$
Proof: $\frac{F_{n+2}}{F_{n+1}} = \frac{F_{n+1}}{F_{n+1}} + \frac{F_n}{F_{n+1}} = 1 + \frac{1}{\frac{F_{n+1}}{F_n}}$

Project Goal:

Is there a pattern for the continued fraction of $\frac{F_{n+1}^m}{F_n^m}$?

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Powers of the Golden Ratio

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi = \frac{1+\sqrt{5}}{2} \implies \lim_{n \to \infty} \frac{F_{n+1}^m}{F_n^m} = \phi^m \phi^1 = \{1; \overline{1}\} \qquad \phi^2 = \{2; \overline{1}\} \phi^3 = \{4; \overline{4}\} \qquad \phi^4 = \{6; \overline{1,5}\} \phi^5 = \{11; \overline{11}\} \qquad \phi^6 = \{17; \overline{1,16}\}$$

• L_n is the *n*th Lucas number. $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$

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Theorem

$$\phi^n = \{L_n; \overline{L_n}\}, \text{ n is odd}$$

$$\phi^n = \{L_n - 1; \overline{1, L_n - 2}\}, \text{ n is even}$$

• The convergents of ϕ^n are $\frac{F_{mn+n}}{F_{mn}}$.

Squares

Examples:

•
$$\frac{F_5^2}{F_4^2} = \frac{25}{9} = \{2; 1, 3, 1, 1\}$$

• $\frac{F_6^2}{F_5^2} = \frac{64}{25} = \{2; 1, 1, 3, 1, 1, 1\}$
• $\frac{F_7^2}{F_6^2} = \frac{169}{64} = \{2; 1, 1, 1, 3, 1, 1, 1, 1\}$

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Theorem

$$\frac{F_{n+1}^2}{F_n^2} = \{2; \overbrace{1,1,\ldots}^{n-3}, 3, \overbrace{1,1,\ldots}^{n-2} \}$$

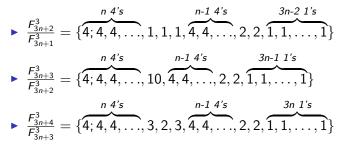
Cubes

Examples:

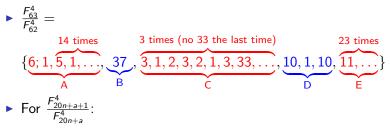
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Cubes

Theorem



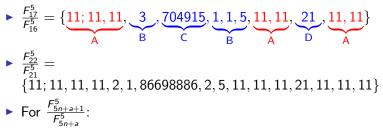
Fourth Power (Conjecture)



- ▶ A consists of 5*n* repititions of 5,1 (the first is 6,1).
- ▶ B varies with *a* mod 4: {1}, {37}, {4,4,33}, {6,1}.
- C consists of *n* repititions of 3,1,2,3,2,1,3,33 (sometimes the start or end is affected by B or D).
- D varies with a mod 5: {31, 1, 9}, {10, 1, 10}, {15}, {33, 3, 1, 2, 2, 1, 1}, {33, 4, 6}.
- E consists of 2^{20n+a+2}/₅ repititions of 11, if a ≡ 2 (mod 5), or 2^{20n+a+2}/₅ 1 repititions of 11, otherwise.

*Red varies in length, while blue varies with a.

Fifth Power (Conjecture)



A's consist of repititions of 11, whose lengths vary with 5n.

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- B's vary with a, but may be compacted into one term, depending on whether n is even or odd.
- D varies with a.

Fifth Power (Conjecture)

- C varies with a, but the value changes.
- For a = 2, the value is exceptionally large.

▶
$$\frac{F_{17}^5}{F_{16}^5} = \{11; 11, 11, 3, 704915, 1, 1, 5, 11, 11, 21, 11, 11\}$$

▶ $\frac{F_{13}^5}{F_{12}^5} = \{11; 11, 10, 1, 46137317, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 9, 11, 11\}$

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In addition, there is a series of 1's rather than 11's.

Future Research

- Fourth and fifth powers
- General theorem
- Polynomial of Fibonacci numbers

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Other Fibonacci-like sequences

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