Staged Self-Assembly

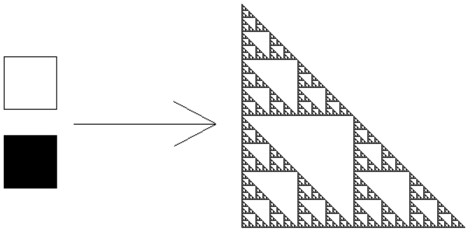
Rohil Prasad   Jonathan Tidor
Second Annual MIT PRIMES Conference

Saturday, May 19, 2012

INTRODUCTION

Definition

Self-assembly *is the process by which order spontaneously forms from simple parts.*



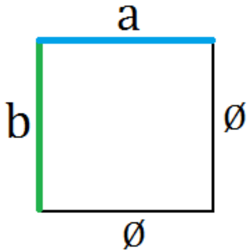
Simple Parts

Definition

A ***tile*** *is a non-rotatable square with a glue on each edge.*

Definition

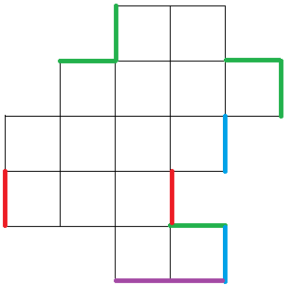
Let G be the set of all ***glues****, including* $\emptyset$*, the* ***null glue****.*



Supertiles

Definition

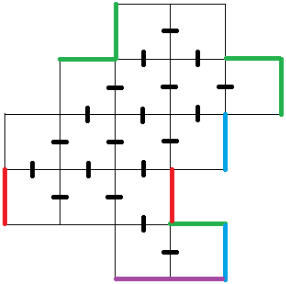
*A **supertile** is a collection of tiles that are bound together. It is said to be **fully connected** if the strength of every bond is non-zero, otherwise, the supertile is **partially connected**.*



SUPERTILES

Definition

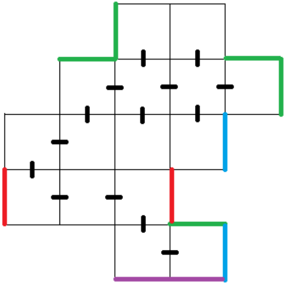
*A **supertile** is a collection of tiles that are bound together. It is said to be **fully connected** if the strength of every bond is non-zero, otherwise, the supertile is **partially connected**.*



SUPERTILES

Definition

A ***supertile*** *is a collection of tiles that are bound together. It is said to be* ***fully connected*** *if the strength of every bond is non-zero, otherwise, the supertile is* ***partially connected****.*



Glues Stick Together

Definition

The ***glue function*** $g : G \times G \to \mathbb{R}_0^+$ *determines the strength of the bond between glues.*

- $g(x, y) = g(y, x) \quad \forall x, y \in G$
- $g(\emptyset, x) = 0 \quad \forall x \in G$

Glues Stick Together

Definition

*The **glue function** $g : G \times G \to \mathbb{R}_0^+$ determines the strength of the bond between glues.*

- $g(x, y) = g(y, x) \quad \forall x, y \in G$
- $g(\emptyset, x) = 0 \quad \forall x \in G$

  For our constructions, we set

- $g(x, y) = 0 \quad \forall x \neq y$
- $g(x, x) \geq 1 \quad \forall x \neq \emptyset$

Glues Stick Together

Definition

*The **glue function** $g : G \times G \to \mathbb{R}_0^+$ determines the strength of the bond between glues.*

- $g(x, y) = g(y, x) \quad \forall x, y \in G$
- $g(\emptyset, x) = 0 \quad \forall x \in G$

  For our constructions, we set
- $g(x, y) = 0 \quad \forall x \neq y$
- $g(x, x) \geq 1 \quad \forall x \neq \emptyset$

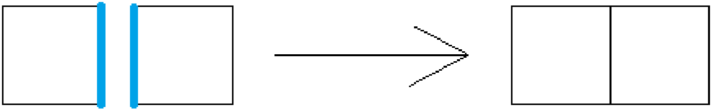
  Typically,
- $g(x, x) = 1 \quad \forall x \neq \emptyset$

Tiles Stick Together

Definition

*The **temperature**, τ, is a property of the system that determines what strength bond is necessary to hold things together.*

- If the strength of the bond between two tiles is at least the temperature, the tiles will connect.

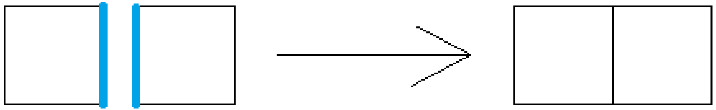


Tiles Stick Together

Definition

*The **temperature**, τ, is a property of the system that determines what strength bond is necessary to hold things together.*

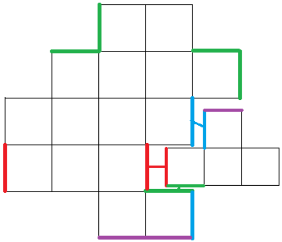
- If the strength of the bond between two tiles is at least the temperature, the tiles will connect.



- Typically we work in temperature $\tau = 1$.

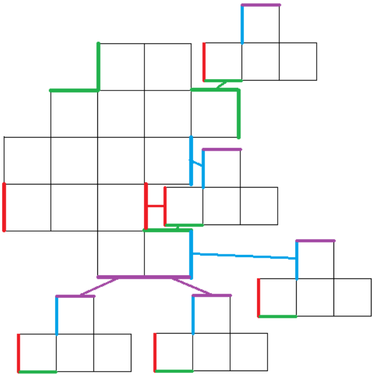
Supertiles Stick Together

- Two supertiles will stick together if the sum of the strengths of the bonds between all adjacent edges is at least the temperature.
- This means that (especially for $\tau = 1$) supertiles can bind together in many ways.



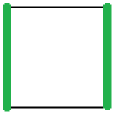
SUPERTILES STICK TOGETHER

- Two supertiles will stick together if the sum of the strengths of the bonds between all adjacent edges is at least the temperature.
- This means that (especially for $\tau = 1$) supertiles can bind together in many ways.



Segment Construction

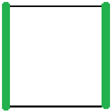
- Consider the single tile:



- It will assemble into the following supertiles:

Segment Construction

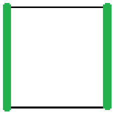
- Consider the single tile:



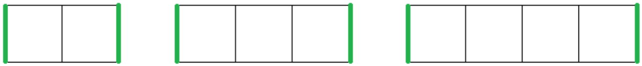
- It will assemble into the following supertiles:

Segment Construction

- Consider the single tile:

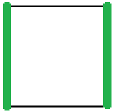


- It will assemble into the following supertiles:

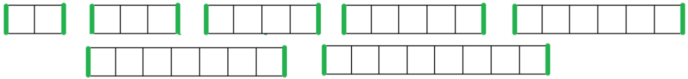


Segment Construction

- Consider the single tile:



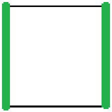
- It will assemble into the following supertiles:



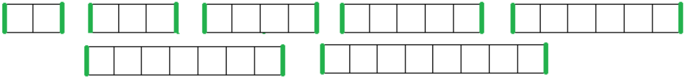
- This set of supertiles is not **uniquely** produced because it continues on indefinitely.

Segment Construction Attempt 1

- Consider the single tile:



- It will assemble into the following supertiles:



- This set of supertiles is not **uniquely** produced because it continues on indefinitely.

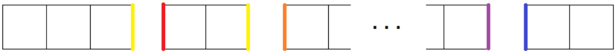
Segment Construction

- Consider the set of tiles:

SEGMENT CONSTRUCTION

- Consider the set of tiles:

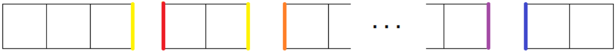
- It will assemble into many supertiles, including:



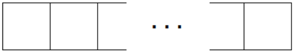
SEGMENT CONSTRUCTION

- Consider the set of tiles:

- It will assemble into many supertiles, including:



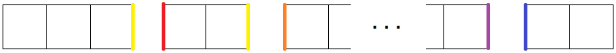
- However, all supertiles continue connecting until they reach their final state as supertile:



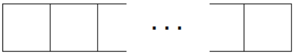
SEGMENT CONSTRUCTION

- Consider the set of tiles:

- It will assemble into many supertiles, including:



- However, all supertiles continue connecting until they reach their final state as supertile:

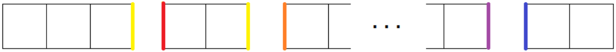


- This supertile is said to be **terminal**, because it cannot bind to any other supertile

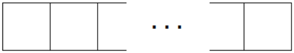
Segment Construction Attempt 2

- Consider the set of tiles:

- It will assemble into many supertiles, including:



- However, all supertiles continue connecting until they reach their final state as supertile:



- This supertile is said to be **terminal**, because it cannot bind to any other supertile

Tile and Glue Complexities

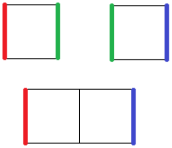
- We want to minimize the number of glues because the creation of a large number of glues provides a technical challenge.

Definition

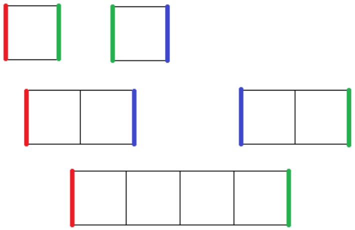
*The **glue complexity** of a construction is $|G|$, the number of glues used, and the **tile complexity**, T, is the number of distinct tiles used in the construction.*

- $|G|/4 \leq T \leq |G|^4$, so we typically attempt to minimize the tile complexity.

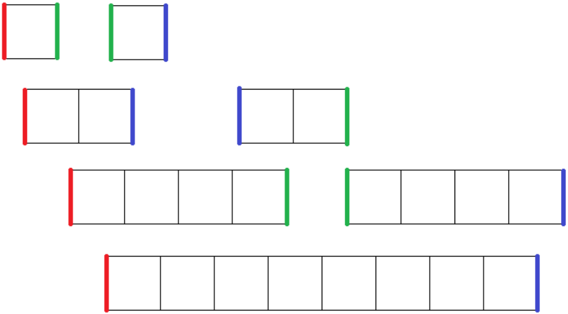
Segment Construction



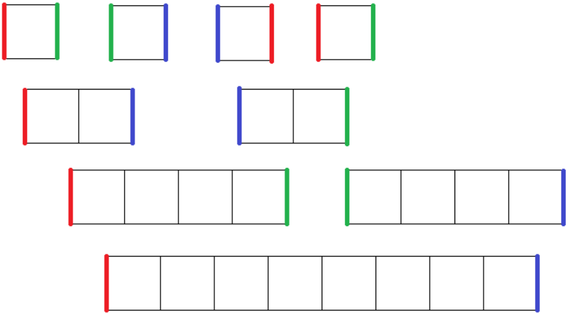
Segment Construction



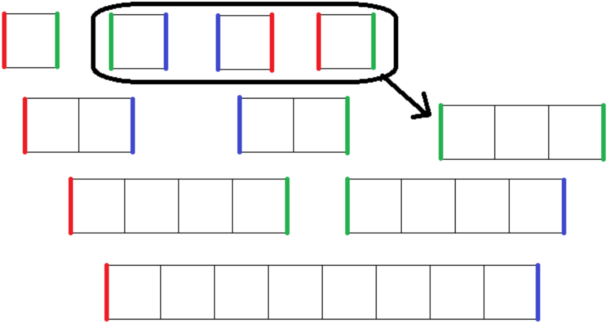
Segment Construction



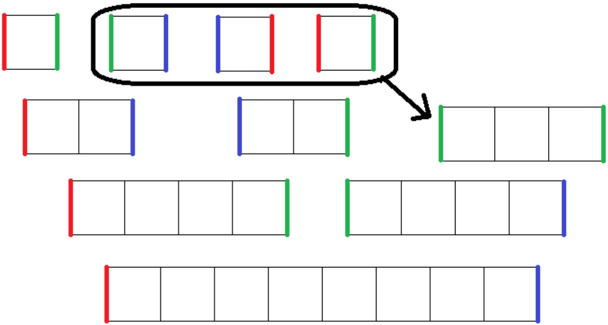
Segment Construction



Segment Construction



Segment Construction Attempt 3



- We need some way of separating groups of tiles so that not every possible connection occurs.

Bins and Stages

Definition

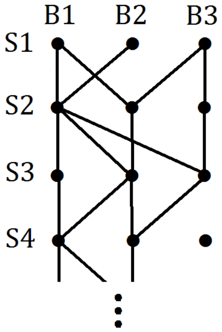
*A **bin** is a container of tiles, and a **stage** is a unit of time.*

- Every stage, the contents of each bin interact until they reach a terminal state.
- Then, the terminally produced supertiles from each bin can be copied and mixed into multiple other bins.
- This mixing can occur between any number of pairs of bins between each stage.
- In addition, specific tiles may be added to each bin at each stage.

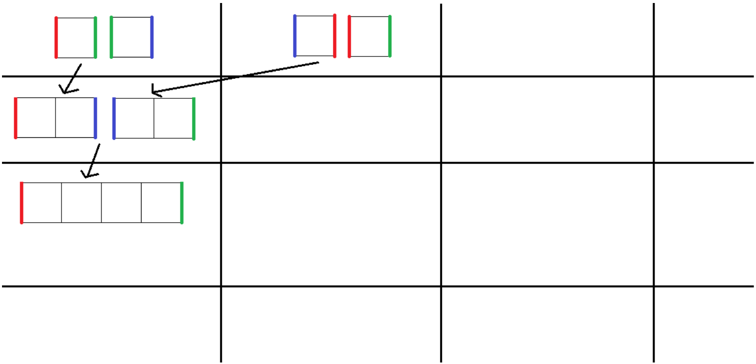
THE MIX GRAPH

Definition

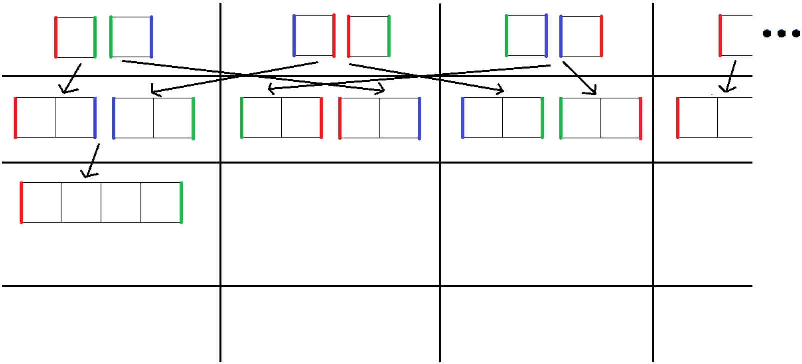
*Given an assembly system with r stages and b bins, the **mix graph** is an rb-vertex graph that provides a visual representation of the mixing of bins from stage to stage.*



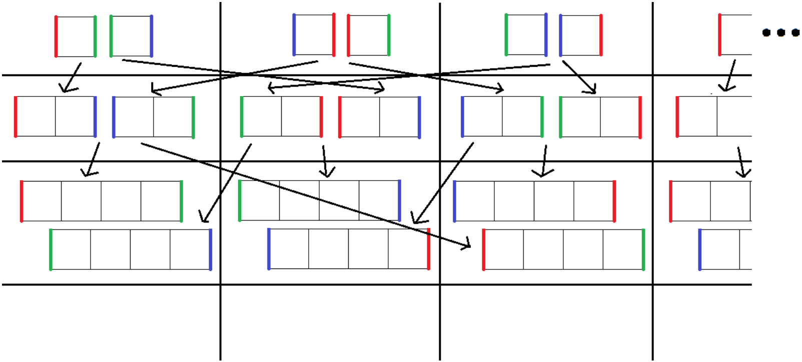
SEGMENT CONSTRUCTION ATTEMPT 3 REVISITED



Segment Construction Attempt 3 Revisited



SEGMENT CONSTRUCTION ATTEMPT 3 REVISITED



Segment Questions

Theorem

A $1 \times n$ line segment can be constructed using 6 tiles, 7 bins, and $O(\log n)$ stages.

Can the same segment be constructed with:

- fewer bins?
- fewer tiles?
- more bins?
- more tiles?

Segment Questions

Theorem

A $1 \times n$ line segment can be constructed using 6 tiles, 7 bins, and $O(\log n)$ stages.

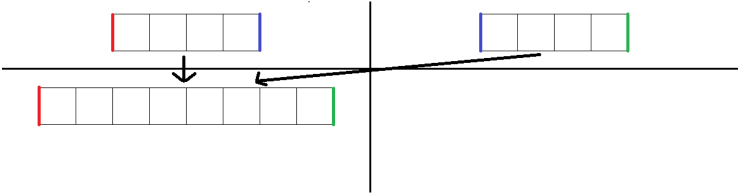
Can the same segment be constructed with:

- fewer bins? yes
- fewer tiles? no
- more bins? yes
- more tiles? yes

$B = 2$

Theorem

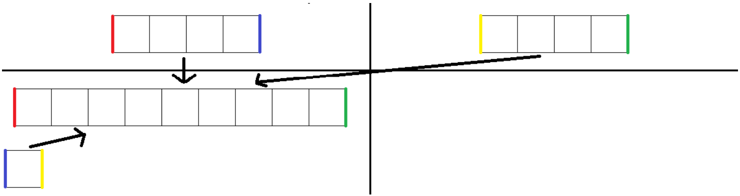
A $1 \times n$ *line segment can be constructed using* $O(1)$ *tiles, 2 bins, and* $O(\log n)$ *stages.*



$B = 2$

Theorem

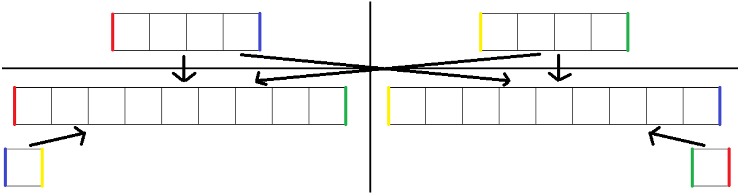
A $1 \times n$ line segment can be constructed using $O(1)$ tiles, 2 bins, and $O(\log n)$ stages.



$B = 2$

Theorem

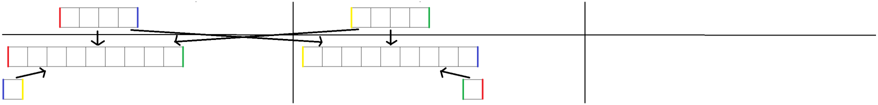
A $1 \times n$ line segment can be constructed using $O(1)$ tiles, 2 bins, and $O(\log n)$ stages.



B Bins

Theorem

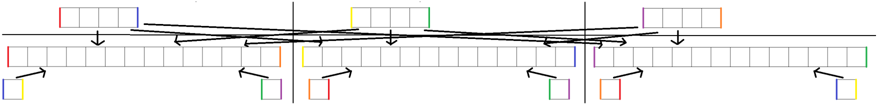
A $1 \times n$ line segment can be constructed using $O(1)$ tiles, B bins, and $O(\log_B n)$ stages.



B BINS

Theorem

A $1 \times n$ line segment can be constructed using $O(1)$ tiles, B bins, and $O(\log_B n)$ stages.



B Bins, T Tiles

Theorem

A $1 \times n$ line segment can be constructed using T tiles, B bins, and $O(\log_B \frac{n}{T})$ stages for $T \geq B$ and in $O(\log_T n)$ stages for $T < B$.

Proof:

- With more tiles, we divide them into separate groups, each with distinct glues, which allows the construction of multiple identical segments in parallel.
- This construction proceeds in $O(\log_B \frac{n}{T})$ stage complexity if there are enough tiles to create a single group of tiles.
- With less than B tiles, we can make one group if we use only T of the bins and leave the others empty.

Is This Optimal?

Is This Optimal?

- Yes it is!
- Given the tiles in our final shape, an analysis of the paths they take in the mix graph gives $\Omega(\log_B \frac{n}{T})$ stages in our construction.

Changing the Temperature

Definition

*Remember that the **temperature**, τ, is the total connection strength along the border of two supertiles that is necessary for connection to occur.*

- When $\tau = 2$, it is useful to have some glues where $g(x, x) = 1$ and some where $g(x, x) = 2$.

Definition

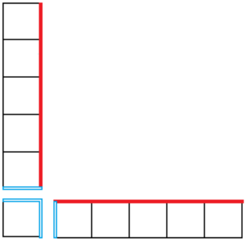
*If $g(x, x) = 1$, x is said to be a **single-strength** glue, while if $g(x, x) = 2$, x is a **double-strength** glue.*

- Using $\tau = 2$ yields simple constructions for shapes that have more complex constructions in $\tau = 1$.

$n \times n$ Square In $\tau = 2$

Theorem

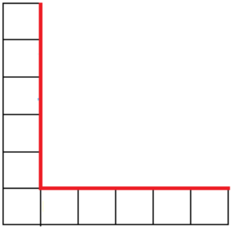
A $n \times n$ square can be constructed using $O(1)$ tiles, 2 bins, and $O(\log n)$ stages in temperature $\tau = 2$.



$n \times n$ Square In $\tau = 2$

Theorem

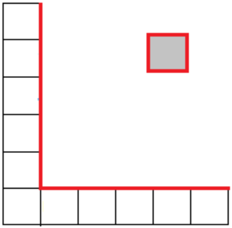
A $n \times n$ square can be constructed using $O(1)$ tiles, 2 bins, and $O(\log n)$ stages in temperature $\tau = 2$.



$n \times n$ Square In $\tau = 2$

Theorem

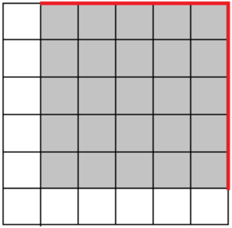
A $n \times n$ square can be constructed using $O(1)$ tiles, 2 bins, and $O(\log n)$ stages in temperature $\tau = 2$.



$n \times n$ Square In $\tau = 2$

Theorem

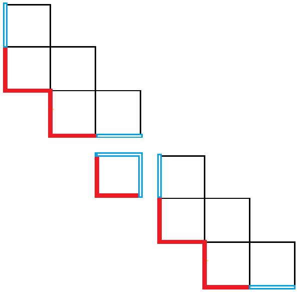
A $n \times n$ square can be constructed using $O(1)$ tiles, 2 bins, and $O(\log n)$ stages in temperature $\tau = 2$.



$n \times n$ Right Isosceles Triangle In $\tau = 2$

Theorem

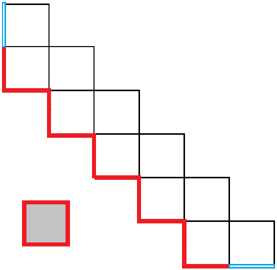
A $n \times n$ isosceles right triangle can be constructed using $O(1)$ tiles, 2 bins, and $O(\log n)$ stages in temperature $\tau = 2$.



$n \times n$ Right Isosceles Triangle In $\tau = 2$

Theorem

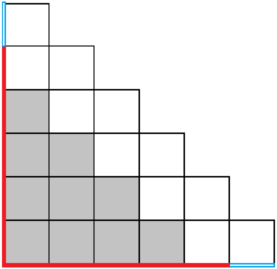
A $n \times n$ isosceles right triangle can be constructed using $O(1)$ tiles, 2 bins, and $O(\log n)$ stages in temperature $\tau = 2$.



$n \times n$ Right Isosceles Triangle In $\tau = 2$

Theorem

A $n \times n$ isosceles right triangle can be constructed using $O(1)$ tiles, 2 bins, and $O(\log n)$ stages in temperature $\tau = 2$.

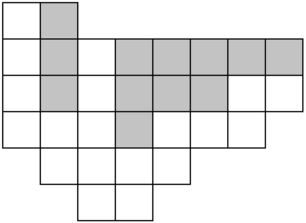


Arbitrary Monotone Shape Construction

Theorem

Any monotone shape can be constructed using $O(n)$ tiles, 2 bins, and $O(\log n)$ stages in temperature $\tau = 2$.

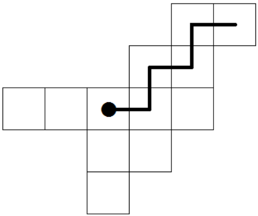
- Construct a 'border' for the desired shape.
- Fill in the other parts of the shape using tiles with the same glue on all sides.

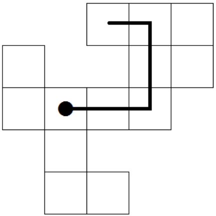


Another Arbitrary Shape Construction

Definition

A shape is called ***radially monotone*** *if, for some choice of the center, every tile can be connected to the center as a path whose lattice distance from the center is increasing.*

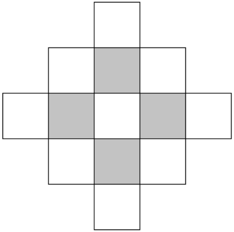




A Diamond Construction

Theorem

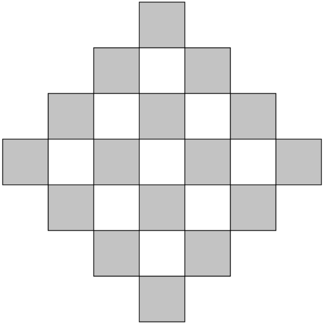
A diamond of radius r can be constructed using $O(r)$ tiles, 1 bin, and $O(r)$ stages.



A Diamond Construction

Theorem

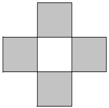
A diamond of radius r can be constructed using $O(r)$ tiles, 1 bin, and $O(r)$ stages.



Arbitrary Radially Monotone Shapes

Theorem

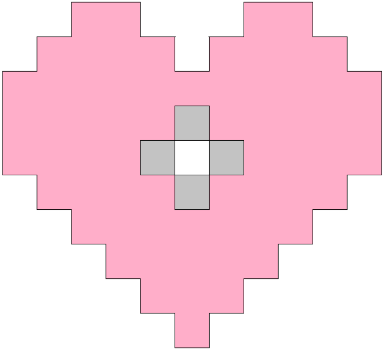
Any radially monotone shape of radius r can be constructed using $O(r)$ tiles, 1 bin, and $O(r)$ stages.



Arbitrary Radially Monotone Shapes

Theorem

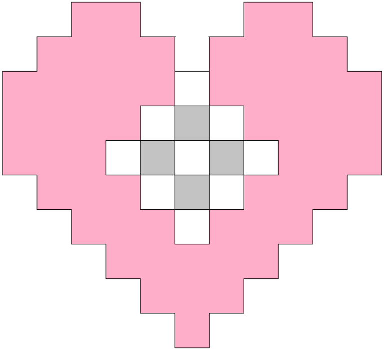
Any radially monotone shape of radius r can be constructed using $O(r)$ tiles, 1 bin, and $O(r)$ stages.



Arbitrary Radially Monotone Shapes

Theorem

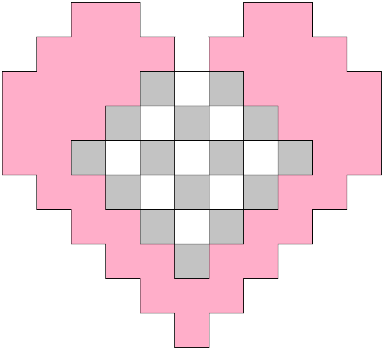
Any radially monotone shape of radius r can be constructed using $O(r)$ tiles, 1 bin, and $O(r)$ stages.



Arbitrary Radially Monotone Shapes

Theorem

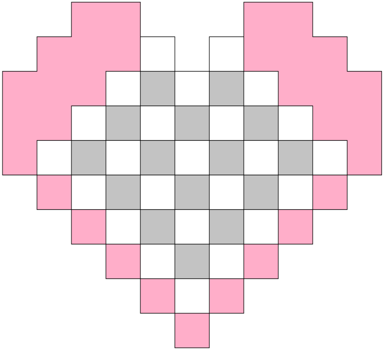
Any radially monotone shape of radius r can be constructed using $O(r)$ tiles, 1 bin, and $O(r)$ stages.



Arbitrary Radially Monotone Shapes

Theorem

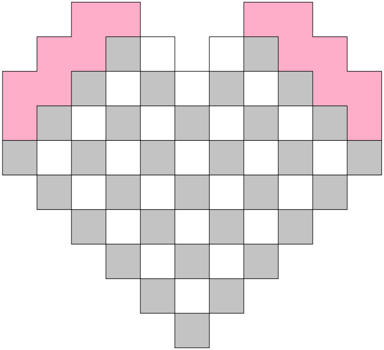
Any radially monotone shape of radius r can be constructed using $O(r)$ tiles, 1 bin, and $O(r)$ stages.



Arbitrary Radially Monotone Shapes

Theorem

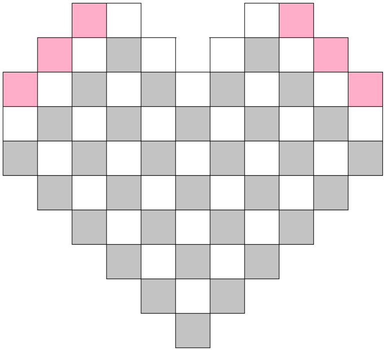
Any radially monotone shape of radius r can be constructed using $O(r)$ tiles, 1 bin, and $O(r)$ stages.



Arbitrary Radially Monotone Shapes

Theorem

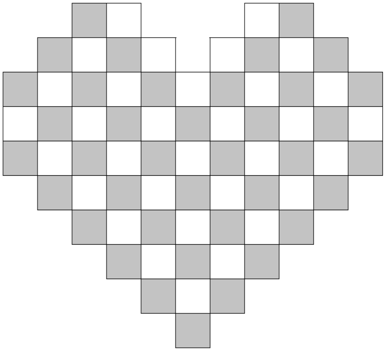
Any radially monotone shape of radius r can be constructed using $O(r)$ tiles, 1 bin, and $O(r)$ stages.



Arbitrary Radially Monotone Shapes

Theorem

Any radially monotone shape of radius r can be constructed using $O(r)$ tiles, 1 bin, and $O(r)$ stages.



Why So Slow?

- Compared to our other constructions, the two constructions of arbitrary shapes have very high tile and stage complexities. Why?

Why So Slow?

- Compared to our other constructions, the two constructions of arbitrary shapes have very high tile and stage complexities. Why?
- From an information theory perspective, an arbitrary shape encodes much more information than a segment or square, which can be described by a single number.
- In fact, using the Kolmogorov Complexity, we can show that these constructions proceed in the optimal stage complexity for their tile and bin complexities.

Further Directions

- Optimize construction of $n \times n$ squares for B bins and T tiles
- Probabilistic model
- Abnormal shapes (Extremely long rectangles, etc.)

ACKNOWLEDGMENTS

We would like to thank:

- Our families for their continual support.
- Jesse Geneson for putting up with us for many months.
- The MIT PRIMES Program for giving us the opportunity to do this research.