The PRIMES-USA 2013 problem set

Dear PRIMES-USA applicant,

This is the PRIMES-USA 2013 problem set. Please send us your solutions as part of your PRIMES-USA application by November 15, 2012. For complete rules, see
http://web.mit.edu/primes/PrimesUSAapply.shtml

Please solve as many problems as you can. You can type the solutions or write them down by hand and then scan them. Please attach your solutions to the application as a PDF file. The name of the attached file must start with your last name, for example, "smith-solutions." Include your full name in the heading of the file.

Please write down not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES-USA.

You are allowed to use any resources to solve these problems, **except** other people's help. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

Note that most of these problems are fairly difficult. We recommend that you do not leave them for the last day, and think about them, on and off, over some time (days and weeks). We encourage you to apply if you can solve at least 40% of the problems. We note, however, that many factors will play a role in the admission decision besides your solutions of these problems.

Good luck!

Problem 1. You toss a coin n times. What is the probability that the number of heads you'll get is divisible by 3? (Find an exact formula, not involving sums of unbounded length; it may depend on the remainder of n modulo 6).

Problem 2. (a) Let $c < 2\pi$ be a positive real number. Show that there are infinitely many integers n such that the equation

$$x^2 + y^2 + z^2 = n$$

has at least $c\sqrt{n}$ integer solutions.

(b) Find a constant $C > 0$ such that there are infinitely many n for which the equation

$$x^5 + y^3 + z^2 = n$$

has $\geq Cn^{1/30}$ nonnegative solutions.

Problem 3. A finite string of 0s and 1s is called **admissible** if it occurs in one of the rows of the Pascal triangle modulo 2. I.e., a string is admissible if it has the form

$$\binom{n}{m}, \binom{n}{m+1}, \ldots, \binom{n}{m+k},$$

where the binomial coefficients are taken modulo 2.

(a) Which strings of length ≤ 4 are not admissible? Why?

(b) Give an explicit description of all admissible strings of length n.

(c) What is the number $a(n)$ of admissible strings of length n? (Write a recursion, guess the answer and prove it by induction; the recursive formula may be different for $n = 2r$ and $n = 2r + 1$).

Problem 4. Positive solutions of the equation $x\sin(x) = 1$ form an increasing sequence $x_n, n \geq 1$.

(a) Find the limit

$$c_1 = \lim_{n\to\infty} n(x_{2n+1} - 2\pi n).$$

(b) Find the limit

$$c_2 = \lim_{n\to\infty} n^3(x_{2n+1} - 2\pi n - \frac{c_1}{n}).$$

Problem 5. Let us say that a polynomial f with complex coefficients is **degenerate** if there exists a square matrix B such that $B \neq f(A)$ for any square matrix A. What are all the degenerate polynomials of degree 2? degree 3? degree 4? any degree?

Problem 6. The sequence $b(n)$ is defined by the recursion

$$b(2n + 1) = 2b(n) + 2,\ b(2n) = b(n) + b(n - 1) + 2,$$

for $n \geq 1$, with $b(0) = 0$, $b(1) = 1$.

(a) Find a generating function for $b(n)$ and deduce a formula for $b(n)$, as explicit as you can.

(b) Let x be a positive real number, and $[a]$ denote the floor (integer part) of a. Find the limit as $m \to \infty$ of $b([2^m x])/[2^m x]$ as a function of x.

Problem 7. Let f be a continuous real function on $[0, \infty]$. Show that if $\lim_{n\to\infty}(f(na)) = 0$ for all $a > 0$ then $\lim_{x\to+\infty} f(x) = 0$.

Note. The final three problems are more difficult than the previous seven. They have more of the flavor of research problems, and are more open-ended. Even when the original problem cannot be solved, partial progress is encouraged, and can take many forms: finding (useful!) ways to model the problem, coming up with reasonable conjectures, identifying the crucial missing steps, and more.

Problem 8. Two countries A and B share a national border which is a straight line. Along this border are $2n$ wells. An architect from each country is chosen to dig n canals within that country. The canals within each country cannot intersect, and each well is the endpoint of exactly one canal in each country.

(a) The architects go home and draw up all the possible plans for canals. How many plans does each architect have?

Once the canals are dug, if one ignores the national borders, the result is a collection of "lakes" (i.e., circular canals). Both countries want lots of lakes, but for political reasons, A wants an even number of lakes, and B wants an odd number. They pay their architects accordingly. If there are k lakes and k is even then architect A earns p^k dollars and B earns nothing. If k is odd then B earns p^k dollars and A earns nothing. For now, assume $p = 2$.

You are architect A. Unfortunately for you, architect B has spies everywhere, and if you settle on a plan in advance, B will counter it. Instead, you decide upon a mixed strategy, i.e. you assign a probability to each plan, and will choose randomly when the time comes. We call a mixed strategy *Nash* if the expected value of the difference in salary (between the two architects) does not depend on B's choice of plan. We call a Nash strategy a *tied strategy* if the expected value of the salary difference is zero.

(b) A spy from country A infiltrates the house of architect B with a mission: find exactly one plan, and destroy it. B will no longer be able to use that plan. It turns out that you now have a unique tied strategy! You may assume this fact, for any plan. You should also consider the assumption granted in part (c).

The spy returns to you and tells you he destroyed a plan where exactly one pair of adjacent wells was attached by a canal. Can you find your unique tied strategy?

(Hint: An exact answer will be difficult, but a recursive formula depending on n is enough. A recursive formula will only be valid if it implies that, for each n, the result is a valid strategy, i.e. that all probabilities are nonnegative and add up to 1)

(c) Unfortunately, architect B was too clever, and had an extra copy of the plan. Just to spite you, B builds precisely the strategy the spy had destroyed. Which architect is expected to benefit most? You may assume that the answer to this question is the same for any plan.

(d) Continue the setup of part (b). Suppose that $p = 1$. Find an n for which multiple tied strategies exist, and demonstrate them. Suppose that $p = \sqrt{2}$. Find an n such that there are multiple tied strategies.

(e) (continued). For each n find a finite list of values of p for which, outside of that list, tied strategies in the setup for (b) are guaranteed to be unique.

(Hint: Try to solve part (b) again for small n, with $p = 2.5$, $p = 3.3333$, $p = 4.25$. None of these are in your list, however.)

(f) Let $p = 2$. Spies inform you that architect B has recovered his missing plan, and has every plan available. Is there a Nash strategy? Which architect does it benefit?

(Hint: First, answer this question - is there a weighting of A's plans, by "probabilities" which need not lie between 0 and 1, such that the weighted average for each of B's plans is equal? Now, what would you need to do to show that each weight can be chosen between 0 and 1? How can you use your answer to part (b)?)

Problem 9. Elsie and Fred are playing SHUFFLE DUEL. In this game, two standard 52-card decks are shuffled: the "playing" deck and the "target" deck. Both decks are visible. After being shuffled, the target deck is then laminated and mounted on the wall, and is not touched again.

The players alternate turns (Elsie goes first because she is older, Fred goes last). On your turn, you must take two adjacent cards in the playing deck, and switch them. Once a certain configuration of the playing deck has been reached, it can never be repeated! You win if the playing deck and the target deck are the same after you have made your move. If your opponent makes a move and you can prove that it is impossible to reach the target deck without passing through a repeated configuration, you may challenge your opponent, and your opponent loses. If the playing deck and target deck are equal to begin with, the last player wins.

a) Elsie and Fred are both very experienced, perfect players. How can you tell who will win?

b) What is the probability that Elsie will win, and why?

Elsie and Fred are "interest gamblers." That is, before the game they put \$ 1 in a bank account. Every time they take a turn, the bank pays interest, multiplying the value of the account by q. When someone wins, they get to cash out from the bank. (Examples: If the game takes 0 turns, Fred wins \$ 1. If the game takes 1 turn, Elsie wins \$ q. If the game takes 2 turns, Fred wins \$ q^2)

Of course, the bank knows that games of SHUFFLE DUEL can take a very long time, so it refuses to pay out unless the game reaches a successful conclusion (the playing deck equals the target deck), and unless the game was AS SHORT AS POSSIBLE for a successful game, given the initial shuffling. Let us call this new game QUICK SHUFFLE DUEL: players may only make moves that are on a shortest path to the target deck.

c) (Easier) What is the expected length of the game? (Harder) What is the expected value of the bank account, at the time of payoff? (Write an explicit product formula).

(Hint: You needn't solve for the 52-card deck. Solve for smaller decks then come up with a formula. Make sure that the expected value is \$ 1 when $q = 1$.)

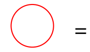
d) Suppose that Ginger and Harold and Ivana and John and Kelly want to play QUICK SHUFFLE DUEL as well. The play rotates between the 7 players. Which player is most likely to win? What about 10 players? At what number of players will the behavior change?

(Hint: Look more closely at your formula from part (c).)

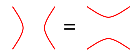
Problem 10. Fix a positive number n. Consider a planar graph where each edge is labelled with a number in $\{1, 2, \ldots, n\}$. This graph need not be connected, and may have connected components which are circles with no vertices. There are only two kinds of vertices in this graph: stars and crosses. A star has 6 incoming edges whose labels alternate $k, k+1, k, k+1, k, k+1$ for some $k \in \{1, 2, \ldots, n-1\}$. A cross has 4 incoming edges whose labels alternate i, j, i, j for some $i, j \in \{1, 2, \ldots, n\}$ with $|i - j| > 1$.

We place an equivalence relation on the set of such graphs, where two graphs are equivalent if they are related by a sequence of the following moves (which can be performed in reverse as well).

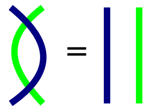
(1) (Circle Removal) Circles with empty interior may be removed.



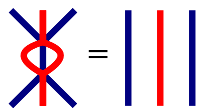
(2) (Bridging) Adjacent edges of the same label can be altered.



(3) (Double Vertex Removal) If two crosses are connected by two edges, both crosses can be removed.

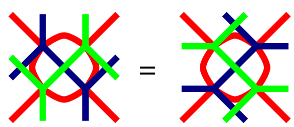


Similarly, if two stars are connected by three edges, both stars can be removed.



(4) (Cross Crossing) A sequence of crosses can be slid to the other side of a cross or star (for any consistent labels).

(5) (Dream Catchers) The following transformation is allowed. In this picture, blue represents the label k, red the label $k+1$, and green the label $k+2$.



Prove that every graph is equivalent to the empty graph.

(Hints: Use induction on n. Given $k \in \{1, 2, \ldots, n\}$ you might consider the k-*subgraph*, consisting only of the edges labelled k. If k is never involved in a star, what does the k-subgraph look like? If k is involved in stars, what does the k-subgraph look like? How might you make the k-subgraph disappear?)