LINEAR EXTENSIONS OF ACYCLIC ORIENTATIONS

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Abstract. Given a graph, an acyclic orientation of the edges determines a partial ordering of the vertices. This partial ordering has a number of linear extensions, *i.e.* total orderings of the vertices that agree with the partial ordering. The purpose of this paper is twofold. Firstly, properties of the orientation that induces the maximum number of linear extensions are investigated. Due to similarities between the optimal orientation in simple cases and the solution to the Max-Cut Problem, the possibility of a correlation is explored, though with minimal success. Correlations are then explored between the optimal orientation of a graph G and the comparability graphs with the minimum number of edges that contain G as a subgraph, as well as to certain graphical colorings induced by the orientation. Specifically, small cases of non-comparability graphs are investigated and compared to the known results for comparability graphs. We then explore the optimal orientation for odd anti-cycles and related graphs, proving that the conjectured orientations are optimal in the odd anti-cycle case. In the second part of this paper, the above concepts are extended to random graphs, that is, graphs with probabilities associated with each edge. New definitions and theorems are introduced to create a more intuitive system that agrees with the discrete case when all probabilities are 0 or 1, though complete results for this new system would be much more difficult to prove.

1. Introduction

In real-world problem modeling, a given population can be modeled as a collection of vertices of a graph. Two members of this population can be connected by an edge if, after some initial point in time t_0, they were in the same place. Suppose now that a disease entered the population at time t_0 and that, with time, it afflicted all members. We would like to know the most likely order (in time) in which the members got the disease. If we know, for each meeting, which of the two members contracted the disease first, we can orient the edge towards the member who contracted it the latest. If this is done for all meetings, an acyclic orientation of the graph is produced. For each acyclic orientation, a linear extension provides a total ordering of all members. Hence, the most likely orientation is the one with the maximum number of linear extensions. For this paper, we will be investigating the properties of this optimal orientation.

Firstly, we will rigorously define the main vocabulary to be used for the rest of this work.

Definition 1. *An* orientation *of an undirected graph $G(V, E)$ is a directed graph formed by assigning a direction to each edge of G. Such an orientation is said to be* acyclic *if there do not exist vertices $v_1, v_2, \cdots, v_k$ with $k \in \mathbb{P}$ such that there is an edge directed from v_i to v_{i+1} for all $1 \leq i \leq k$ (with $v_{k+1} = v_1$).*

Definition 2. *A* directed path of length k *in an orientation of a graph $G(V, E)$ is a set of edges $e_1, \cdots, e_k$ for which each pair of consecutive edges are adjacent, and where the sink vertex of one edge is the source vertex of the next.*

Any acyclic orientation of a graph $G(V, E)$ induces a partial ordering $<$ on the set of vertices of G, defined so that for two vertices v_1, v_2, we have $v_1 < v_2$ if and only if there is a directed path from v_1 to v_2 along the edges of the orientation.

Definition 3. *A* linear extension *of an acyclic orientation is a total ordering $<_T$ on the set of vertices such that if $v_1 < v_2$ in the partial ordering, then $v_1 <_T v_2$.*

For most acyclic orientations of most graphs, linear extensions are not unique. As mentioned, given an undirected graph $G(V, E)$, we would like to determine the acyclic orientation of G that maximizes the number of linear extensions. For certain graphs, this result is known. I will now present these known results and brief outlines of their proofs, all of which come from [Gir14].

Theorem 1. [Sta88] *For a bipartite graph $G(V, E)$, the orientations that maximize the number of linear extensions are exactly the bipartite orientations, that is, the orientations with no directed paths of length* 2.

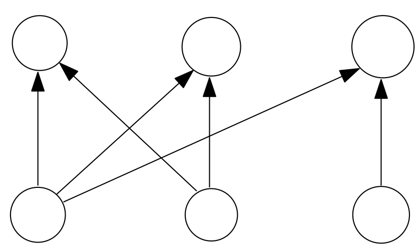

Figure 1. An example of a bipartite orientation

Proof. The main idea of this proof is to fix a bipartition of the graph and an orientation where all directed edges point from one side of the bipartition, called the *source side*, to the other side of the bipartition, called the *sink side*. This type of orientation is also know as *bipartite orientation*. Firstly, we start with an arbitrary labeling of the $n := |V|$ vertices with the set of integers between 1 and n, so that no two vertices are assigned the same integer, and we annotate down the acyclic orientation obtained from directing every edge from the smaller number to the larger number. Then, a set B is constructed that consists of the vertices whose adjacent directed edges differ in orientation from those of the bipartite orientation. If B is the empty set, then the orientation is bipartite and we are done. Otherwise, B has a maximal element on the source side and a minimal element on the sink side. Swapping the labels of these two vertices and re-calculating the orientation of the graph according to this re-labelling reduces the number of elements of B, and it can be shown that the number of linear orientations of the resulting orientation is not less than the original number of linear orientations. This process can then be repeated until B is the empty set, at which point the orientation is bipartite, and since the number of linear extensions is non-decreasing, the bipartite orientation maximizes the number of linear extensions. □

Using this result, we can find the orientation that maximizes the number of linear extensions for odd cycles as well. Here is another proof outline.

Theorem 2. *For an odd cylce $G(V, E)$, the orientations that maximize the number of linear extensions are exactly the orientations with exactly one path of length 2.*

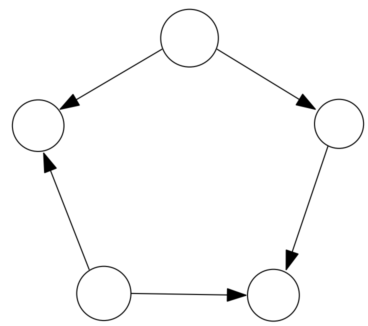

FIGURE 2. An optimal orientation for the odd cycle on five vertices

Proof. Since G is an odd cycle, in every acyclic orientation of G there must exist at least one directed path of length 2, since otherwise G would be bipartite. Let one such path in G connect vertices v_1, v_2, v_3 such that there is an edge that goes from v_1 to v_2 and an edge that goes from v_2 to v_3. An effective edge can then be added that goes from v_1 to v_3, and if the two edges between v_1, v_2, and v_3 are replaced by this effective edge, the result is a bipartite graph. It is known by the previous theorem that the maximum number of linear orientations for such a bipartite graph is a bipartite orientation with no paths of length 2, and this can be extended back to the original graph, so there would be exactly one path of length 2 for an orientation that maximizes the number of linear extensions. □

The techniques needed for the remaining result are a bit more involved, so for brevity I will simply present the result without a proof. Details can be found in [Gir14]. Firstly, we need to define two more terms.

Definition 4. *A* comparability graph *is a graph such that there exists a partial ordering $<$ on the set of its vertices V, and there is an edge between vertices v_1 and v_2 if and only if either $v_1 < v_2$ or $v_2 < v_1$.*

Definition 5. *An orientation of a comparability graph G is called* transitive *if in the order implied by the orientation, two vertices are comparable if and only if there is an edges between them in G.*

Theorem 3. [Gir14] *For a comparability graph G, the orientations of G that maximize the number of linear extensions are exactly the transitive orientations of G.*

These results and their consequences are the main known tools for finding the optimal orientations of any given graph, and we will later exploit these tools to prove novel results.

What follows is an introduction to the Max-Cut Problem, definined below. The motivation for including the description of this problem are the similarities between its solutions and optimal orientations, specifically in the case of bipartite graphs and odd cycles. These similarities led to the formation of Conjecture 1, also discussed below.

2. The Max-Cut Problem

Firstly, following the discussion in [Gir14], I will present what is known as the Max-Cut Problem for a graph G, and then explain how it relates to the problem of finding the orientations of G that maximize the number of linear extensions.

Problem 1. *Given a graph $G(V, E)$, determine a partition of V into two classes such that the number of edges between these two classes is maximized.*

This problem is known to be, in general, NP-Complete; however, for planar graphs, it can be solved in polynomial time. The following conjecture is aimed at relating the Max-Cut Problem to the problem of finding the orientation that maximizes the number of linear extensions.

Conjecture 1. *Given a graph $G(V, E)$ and a solution to the max-cut problem, there exists an orientation of G that is bipartite with respect to the two blocks of this max-cut solution, and that also maximizes the number of linear extensions.*

The converse is not necessarily true — that is, given an orientation of G that maximizes the number of linear extensions, there does not necessarily exist a solution to the Max-Cut Problem such that the orientation is bipartite with respect to it. I present a simple counter-example.

Example 1.

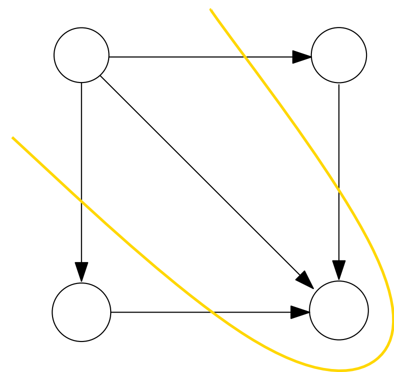

FIGURE 3. A counter-example to the converse of Conjecture 1

Consider the above oriented graph. The orientation is transitive, and so by Theorem 3, maximizes the number of linear extensions. The above partition (consisting of the two sets of vertices on each side of the curve) has 4 edges going through it, and since the graph is not bipartite, any solution to the Max-Cut Problem for this graph has four edges. However, it can be checked that this orientation is not bipartite for any such partition since there are two paths of length 2, and so the converse of the conjecture is not necessarily true.

Clearly the conjecture is true for bipartite graphs. It is also true for odd cycles. Suppose an odd cycle has $2n+1$ edges. Then it can be easily seen that the solution to the Max-Cut Problem for this graph has $2n$ edges between the two partitions. Assigning orientations to each of these $2n$ edges such that the orientation is bipartite results in the bipartite orientation identical to the one in the proof of Theorem 2. The final edge can be oriented either way and the result would still be that the orientation has the maximum number of linear extensions.

Unfortunately, after further investigation, the conjecture was proven to be false. I present here the construction, since it is simple, ingenious, and relevant for the disussion.

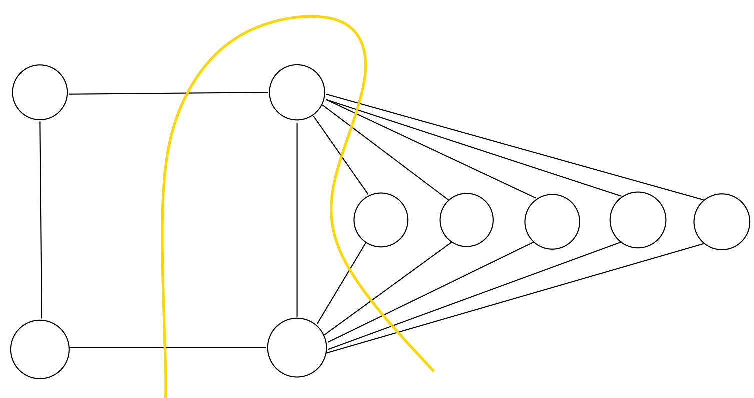

FIGURE 4. A counter-example to Conjecture 1

Consider the above graph and the given solution to its Max-Cut Problem. If this conjecture is to be true (that is, for an optimal orientation to be bipartite with repsect to this solution), the two edges not intersected by the given curve must both be oriented in the same direction. However, this will always induce a new edge in as a diagonal of the square, no matter which way the two other edges of the square are oriented. Therefore, any orientation bipartite to the given solution cannot be transitive. Since the original graph was a comparability graph, by Theorem 3, any such orientation cannot maximize the number of linear extensions, so Conjecture 1 is false in general. We note that, actually, even though the solution to our problem is well-understood for comparability graphs, the solution to the max-cut problem for them is unknown.

3. Induced Colorings

Here, I will introduce the idea of associating a coloring of a graph with an acyclic orientation. Consider an acyclic orientation of a graph $G(V, E)$. For each vertex, there is a set of directed paths associated with that vertex that consists of all directed paths whose final edge is directed into that vertex. Each vertex is then assigned a number equal to the maximum length of any paths in the set. These numbers then partition the vertices of G, and this partition determines a coloring of G.

Theorem 4. *This coloring induced by the orientation is a proper coloring.*

Proof. Suppose, on the contrary, that there are two vertices v_1 and v_2, each with the same value p, that are connected by an edge e. Since both vertices have a value p, for both vertices, there exists a path of length p. Since the orientation is acyclic, these two paths can go through v_1 and v_2 at most once. Now suppose, without loss of generality, that e is directed from v_1 to v_2. The maximum path of v_1 cannot pass through e, since that would mean it passes through v_1 then returns to it, impossible for an acyclic graph. Therefore, there exists a path of length p that ends at v_1 and does not pass through e. However, adding e to this path results in a path of length $p+1$ that ends at v_2, contradicting the assumption that the value of v_2 was p. Therefore, no two vertices in the same partition can have a connecting edge, so the coloring is proper. □

In fact, this is part of the Gallai-Hasse-Roy-Vitaver Theorem [Roy67], which states that, over all acyclic orientations of a graph, the minimum length of the maximum directed path is equal to the minimum number of colors with which the graph can be properly colored.

With an induced proper coloring now defined, it becomes a natural question to ask when this proper coloring is a minimal proper coloring.

Conjecture 2. *Suppose a graph $G(V, E)$ has an acyclic orientation that maximizes the number of linear extensions. This orientation also induces a minimal proper coloring.*

This conjecture is easily shown to be true for both bipartite graphs, odd cycles, comparability graphs, and some minimal non-comparability graphs that we would like to study.

Here now is a second conjecture that may be related to the above conjecture, though a definite correlation has not yet been found.

Conjecture 3. *Given a graph $G(V, E)$, an acyclic orientation that maximizes the number of linear extensions also minimizes the number of edges in its induced comparability graph, i.e. the graph obtained by adding an edge between every pair of comparable vertices in the orientation. Hence, from the set of all $\binom{|V|}{2}$ pairs of vertices, the orientation minimizes the number of comparable pairs among all acyclic orientations of G.*

By Theorem 3, this is true for comparability graphs. This is also true for odd cycles. By Theorem 2, an orientation of an odd cycle has a maximum number of linear extensions if and only if it has exactly one path of length 2, and so if and only if the comparability graph has exactly one more edge than the cycle itself. Since odd cycles are not comparability graphs, the comparability graph for any orientation must have at least one more edge than the cycle, and so these orientations do indeed minimize the number of edges in the comparability graph.

Perhaps, I thought, an easier approach than dealing with these two conjectures separately is to show that minimal proper colorings are directly related to orientations with minimal number of edges in the comparability graph, in the following sense:

Conjecture 4. *Given a graph $G(V, E)$, any acyclic orientation that minimizes the number of edges in the comparability graph, also induces a minimal proper coloring.*

However, this conjecture is false, as shown below.

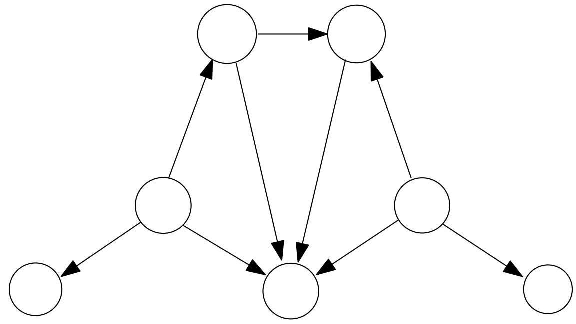

FIGURE 5. Counter-example to Conjecture 4

The given orientation of the above graph — call it G — only induces a single edge, and since it is not a comparability graph, this is the minimum possible number of edges induced by the comparability relation in an acyclic orientation. However, the coloring of the graph induced by this orientation has four colors, while the graph itself is three-colorable, meaning Conjecture 4 is false in general.

Example 2.

Let us investigate G now in further detail. Several interesting behaviors that differ from those of a comparability graph are naturally evidenced in G, so it will be convenient to explore in greater detail this graph in relation to our problem. Examples of these unique behaviors are the following: G is the simplest graph for which the optimal orientation is not known from a proof but only from exhaustive computation; and moreover, this graph already disproves Conjecture 4.

Let us work on the enumerative problem of computing the number of linear extensions of two different acyclic orientations of G. Label the vertices as follows: a is the vertex with degree 4, then b, c, d, e clockwise around the five-cycle. Of the remaining vertices, the one adjacent to b is labeled f and the one adjacent to e is labeled g.

Firstly, let us begin with the acyclic orientation of G that was introduced above for disproving 4. To determine the exact number of linear extensions for this orientation, the edge connecting c and d was removed. Then, since vertex a is larger than its four adjacent vertices, it can only be associated with the values $5, 6$, or 7 in any linear extension.

First, suppose a has value 5. Then vertices b, c, d, and e must have values $1, 2, 3$, and 4 in some order, and the two antennae then must have values 5 and 6. Which antenna has which value does not matter, since both are maximums in every case. This case therefore reduces to assigning values to b, c, d, and e. Since assigning some four values to b and c uniquely determines the remaining values, there are $\binom{4}{2} = 6$ ways to assign these values, and so $2 \cdot 6 = 12$ linear extensions for which a has a value of 5.

Now suppose a has value 6. Then one of the antennae must have value 7, so suppose without loss of generality that the one attached to e has a value 7, and in the end we will multiply by 2 to account for this. Then, if the second antenna has value 5, similarly to above there will be $\binom{4}{2} = 6$ linear extensions. If it has value 4, then c or d must have value 5. If c has value 5, then by similar logic to above the value of b uniquely determines the remaining values, so there are 3 such cases. Similarly if d has value 5 then there are 3 cases, so there are 6 total cases if the second antenna has value 4. If the second antenna has value 3, then the value of b must be either 2 or 1. If b is 2, then a can be 4 or 5, and the remaining vertices are determined, for 2 cases. If b is 1 then a can be $2, 4$ or 5, and the remaining vertices are determined, for 3 cases. Therefore, if the second antenna has value 3, then there are $2 + 3 = 5$ cases. Finally, suppose the second antenna has value 2 (it cannot have value 1 since it is greater than b). Then the value of b must be 1, and from this the value of c completely determines the other vertices. Since c can be $3, 4$, or 5, there are 3 cases here. In total, if a has value 6, there are $2(6+6+5+3) = 40$ cases.

Finally, suppose a has value 7. Since a now has the maximum value, it can be removed from the graph, leaving two disconnected paths of length 2. For the path containing b and c, we can pick $\binom{6}{3} = 20$ of the numbers to be in that path. The maximum of these numbers must be at b; otherwise, the other two can be in any order, so there are $2 \cdot 20 = 40$ ways to order the elements on one side. Similarly, there are 2 ways to order the minimal elements on the other side, for a total of $40 \cdot 2 = 80$ cases when a has a value of 7.

In total, if the top edge is removed, there are $12+40+80 = 112$ linear extensions. If the top edge is returned, by the symmetry of the orientation, there are equal numbers of linear extensions for both ways it is oriented, so if we choose one direction for it to be oriented, there are $\frac{112}{2} = 66$ linear extensions.

Now I will count the number of linear extensions of a slightly different orientation. Take the same orientation as above, but with the edge between b and c oriented towards b, and the antenna from b oriented towards b. Casework similar to that above is needed.

Firstly, suppose a has value 6. Then, since the antenna from e is the only vertex not less than a, it must have value 7. This leaves a path with 5 vertices alternating up and down. The number of cases here is then known to be the fifth Euler number (also known as the up/down numbers), which is 16.

Then, the only remaining case is when a has a value of 7. This produces a path, this time with 6 vertices, again alternating up and down. The number of cases is then the sixth Euler number, known to be 61.

In total, then, for this orientation, there are $16 + 61 = 77$ linear extensions. It can then be seen that this is the maximal possible number of linear extensions of any acyclic orientation of G. However, this orientation is not natural for G in any obvious way. □

Motivated by the new challenges presented in Example 2, the main scope for the remainder of our efforts will be to study these conjectures exclusively for minimal non-comparability graphs. As suggested from our words, one such example of a minimal non-comparability graph is precisely the graph presented in Example 2, which in fact belongs to a larger infinite class of minimal non-comparability graphs. All the classes of minimal non-comparability graphs, finite and infinite, were discovered in a fundamental work of Tibor Gallai, not available in English as of now. However, descriptions of these graphs can be found in [TJMJ76].

4. Odd Anti-cycles and Related Graphs

For ease of communication, suppose that our graphs hereon have n vertices and have these vertices positioned on a circle so that they form a regular n-gon. In this section, we will compute the maximal number of linear extensions of both odd cycles and anti-cycles.

To begin, for the sake of completeness, I will find the maximum number of linear extensions for odd cycles.

Definition 6. *Let $k \in \mathbb{P}$. The k-th* Euler number*, or* up-down number*, is the total number of permutations $(a_1, a_2, \ldots, a_k)$ of the set $\{1, 2, 3, \ldots, k\}$ such that $a_1 > a_2 < a_3 > a_4 < \ldots$.*

Theorem 5. *Given an odd cycle on n vertices, the maximum number of linear extensions over all acyclic orientations is $\frac{1}{2}E_n$, where E_n is the n-th Euler number.*

Proof. By Theorem 2, the maximum number of linear extensions for an odd cycle occurs when the edge directions alternate clockwise and counterclockwise, and there is exactly one two-directed path. Removing one edge of the two-directed path yields a path graph on n vertices with an orientation where edge directions are alternating. By a simple inspection, the number of linear extensions of this acyclic orientation of the n-path graph is E_n. The removed edge compares the two end-vertices of the path graph.

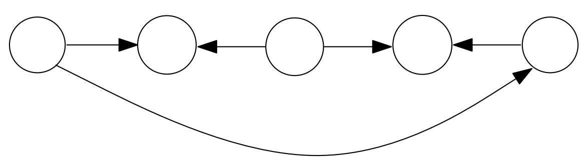

Figure 6. Enumerating Linear Extensions on an Odd Cycle

By the symmetry of this acyclic orientation, and since n is odd, we can *flip* each linear extension of this orientation, so that the number corresponding to the k-th vertex from one end is now the number corresponding to the $(n+1-k)$-th vertex from that same end of the path. This *flip* labelling is also a linear extension of the same acyclic orientation. We can therefore pair each extension with its *flip*, so that for any pair, the largest end-vertex in one linear extension will be the smallest end-vertex in its partner. Therefore, a given end-vertex is larger than the other in exactly half of the linear extensions, so the number of linear extensions for an odd cycle is $\frac{1}{2}E_n$. □

Continuing, an *odd anti-cycle* is the complement of a cycle with an odd number of vertices. Odd anti-cycles are also examples of minimal non-comparability graphs.

Define a **k-removed graph** on n vertices to be the resultant graph when exactly the edges with the k smallest lengths are removed from the complete graph K_n, and where edge lengths are measured in our pictorial representation of the graph in which vertices form a regular n-gon. Then, the 1-removed graph on n vertices with n odd is an odd anti-cycle, and a $\frac{n-3}{2}$-removed graph on n vertices, again with n odd, is an odd cycle. For

$k \geq 1$, by definition, no consecutive vertices of the k-removed n-gon will be connected, since these constitute the smallest edges.

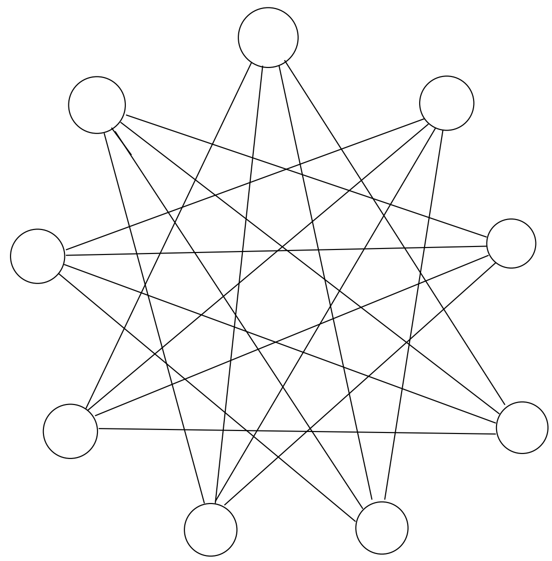

FIGURE 7. A 2-removed graph on 9 vertices

Theorem 6. *For any integer k with $\frac{n-3}{2} \geq k \geq 1$, with n odd, any acyclic orientation of a k-removed graph on n vertices has two comparable vertices that are consecutive in the regular n-gon.*

Proof. Suppose, on the contrary, that there are no such pairs of comparable vertices. Consider the edges connected to some arbitrary vertex. If there are two consecutive diagonals of the n-gon, one directed in and one directed out of our vertex, then this induces a comparability relation between two consecutive vertices of the n-gon, contradiction. However, if, for any vertex, there exists at least one edge directed in and one directed out of it, then there must exist two consecutive edges in opposite directions. Therefore, every vertex must be either a maximum or a minimum. Now consider a line of symmetry passing through one vertex. By construction and since the graph is connected (since it contains a spanning odd cycle as subgraph), there must exist two edges connecting the vertex passing through the line of symmetry to those two closest to the line of symmetry but not on it. If the vertex on the line of symmetry is a maximum, then these edges are directed away from the two vertices closest to the line of symmetry, so those two vertices must be minimums, and therefore are the same type of extremum. Similarly, if the vertex on the line of symmetry is a minimum, then the two vertices closest to the line of symmetry are both maximums, and are again the same type of extremum. Since this works with every line of symmetry, any two consecutive vertices of the n-gon are the same type of extremum, so every vertex of the graph is of the same type. However, we cannot have an acyclic orientation of a connected graph where every vertex is a maximum or every vertex is a minimum, contradiction. Therefore, there must two consecutive vertices of the n-gon that are comparable. □

In particular, since $k \geq 1$, there are no edges between consecutive vertices of the n-gon, so a k-removed graph on n vertices with $\frac{n-3}{2} \geq k \geq 1$ is not a comparability graph.

Now, define an orientation of a k-removed graph on n vertices (n odd) with $\frac{n-3}{2} \geq k \geq 1$ as follows. Label the vertices in counter-clockwise order, $1, 2, \cdots, n$, and direct each existing edge from the smaller to the larger number. Since the k smallest edges are removed, this means that each vertex does not connect to the k closest vertices on either side of it. Therefore, the vertices labeled 1 through $k+1$ are all minimal, and the vertices labeled $n-k$ through n are all maximal, and there are $k+1$ of each. Call this the *rotary orientation* of the k-removed graph on n vertices.

We will now look at the coloring induced by this rotary orientation.

Theorem 7. *The rotary orientation always induces a minimal proper coloring on a k-removed graph on n vertices.*

Proof. Fix a rotary orientation on the graph. Suppose we have colors $c_0, c_1, \cdots, c_{m-1}$, where for all $i \in \{0, 1, \ldots, m-1\}$, c_i is the color corresponding to a vertex that ends a maximal (by containment of edges) directed path of length i in the orientation. The $k+1$ minimal elements will all be colored c_0. Then, the next

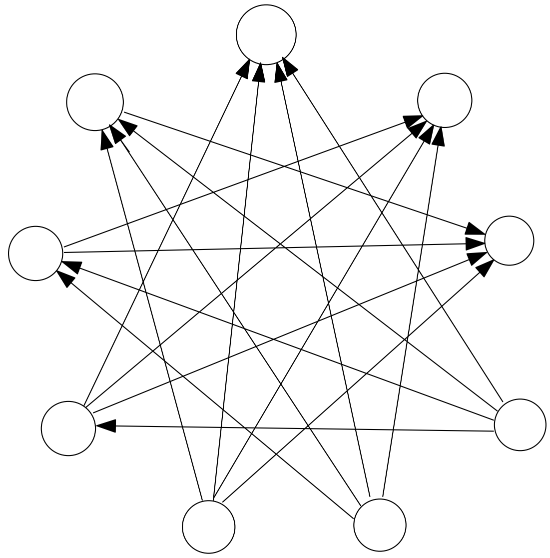

FIGURE 8. A 2-removed graph on 9 vertices with the rotary orientation

$k+1$ vertices in a counter-clockwise direction will, by construction, have their only inward edges coming from those minimal elements, so they are assigned color c_1. Continuing counter-clockwise in this manner, the next $k+1$ vertices will be assigned color c_2, and so on, so since there are n vertices to begin with, the total number of colors is $\lfloor \frac{n}{k+1} \rfloor$. Now suppose there exists a proper coloring with fewer than $\lfloor \frac{n}{k+1} \rfloor$ colors. Then there must be a color assigned to at least $\frac{n}{\lfloor \frac{n}{k+1} \rfloor} > \frac{n}{\frac{n}{k+1}} = k+1$ vertices, so there must be a color assigned to at least $k+2$ vertices. However, any set of $k+2$ vertices has at least $k+1$ different distances between the vertices, and since only k lengths of edges were removed, there must be an edge remaining. This is a contradiction since this creates an edge between two vertices of the same color, and so $\lfloor \frac{n}{k+1} \rfloor$ is the chromatic number of a k-removed graph on n vertices. Therefore, the rotary orientation induces a minimal proper coloring. □

If these orientations are optimal orientations, then this satisfies Conjecture 2. In fact, we can show that for $k = 1$ (that is, for odd anti-cycles), this orientation is indeed optimal.

Theorem 8. *When $k = 1$, the rotary orientation for the k-removed graph on n vertices has the maximal number of linear extensions over all acyclic orientations.*

Proof. From the previous theorem, every orientation on a k-removed graph on n vertices induces a directed edge between two consecutive vertices of the n-gon. For every orientation, adding this induced edge to the graph will not change the linear extensions. Moreover, from the symmetry of the graph, adding any such edge will result in the same graph modulo isomorphism, so there is a bijection between the linear extensions of any acyclic orientation of the k-removed graph on n vertices and the linear extensions of some acyclic orientation of the graph obtained by adding an edge between consecutive vertices. However, we can verify that the rotary orientation induces (by comparability) exactly one edge between the vertices labelled with 1 and n, and so this results in a transitive orientation when that edge is added when $k = 1$. We know from Theorem 3 that exactly the transitive orientations create the maximum number of linear extensions, and so from the bijection, the above orientation will introduce the maximum number of linear extensions for 1-removed graphs on n vertices. □

From symmetry, the same orientation but rotated or reflected will also introduce the maximum number of linear extensions. Interestingly, we can now count the maximum number of linear extensions over all acyclic orientations for odd anti-cycles of n vertices with following theorem.

Theorem 9. *Define the Fibonacci Numbers as $F_0 = F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for integers $k \geq 2$. Then, for an odd anti-cycle with n vertices, the rotary orientation has F_n linear extensions.*

Proof. Since there is an induced edge between the consecutive maximum and minimum vertices, add that directed edge to the graph. The resulting graph then still has the same number of linear extensions as the odd anti-cycle. Now, define a directed even anti-cycle in the same manner as above (*i.e.* consider the complement of an even cycle, then number the vertices in counter-clockwise order, and lastly direct edges according to the numbering), and again add a directed edge between from the vertex with label 1 to the vertex with label n. Now, for three vertices define the corresponding graph to have exactly one directed edge, and for two vertices

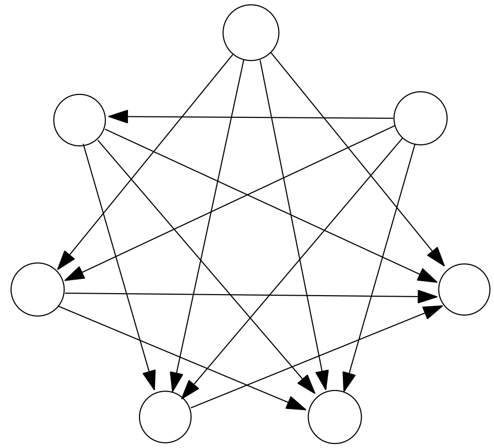

FIGURE 9. The rotary orientation on an anti-cycle of 7 vertices

define the corresponding graph to be the empty graph on 2 vertices. Collectively, call these directed anti-cycles with an extra edge *altered anti-cycles*.

Let $f(n)$ denote the number of linear extensions for the altered anti-cycle on n vertices. We easily have $f(0) = f(1) = 1$, $f(2) = 2$, and $f(3) = 3$. With some casework, we can also find that $f(4) = 5$ and $f(5) = 8$.

Now we proceed inductively. Take an altered anti-cycle with $n \geq 6$ vertices. Either the two minimum vertices correspond to $\{1, 2\}$ in some order, or the vertex with indegree 1 corresponds to 2 and the two minimum vertices correspond to $\{1, 3\}$, with the vertex corresponding to 1 not being adjacent to the vertex corresponding to 2. The same situation arises when dealing with the two maximum vertices and the vertex with outdegree 1. We now separate the linear extensions in four cases.

In the first case, the two minimum vertices correspond to $\{1, 2\}$ and the two maximum vertices correspond to $\{n - 1, n\}$. This can happen in any order, for $2 \cdot 2 = 4$ possibilities. Now remove the four maximum/minimum vertices and any edges that connect to them, and we are left wih an altered anti-cycle on $n-4$ vertices, also with the conjectured configuration. This new graph has $f(n-4)$ linear extensions, so in this case there are $4f(n-4)$ linear extensions.

The case where the two minimum vertices correspond to $\{1, 3\}$ and the two maximum vertices correspond to $\{n-2, n\}$ is similar. In this case, however, there is only one possible positioning for $\{1, 2, 3\}$ and $\{n-2, n-1, n\}$, and removing those six vertices results in an altered anti-cycle with $n - 6$ vertices.

Now, if the two minimum vertices correspond to $\{1, 2\}$ while the two maximum vertices correspond to $\{n - 2, n\}$, then the vertex with outdegree 1 must correspond to the value $n - 1$, while the vertex with indegree 1 has no such restriction. Therefore, we can swap the values 1 and 2 for 2 possibilities. Removing the maximum and minimum vertices as well as the vertex with outdegree 1 results in an altered anti-cycle with $n - 5$ vertices, so the total number of linear extensions in this case is $2f(n-5)$. Similarly, if the two minimum vertices correspond to $\{1, 3\}$ and the two maximum vertices correspond to $\{n - 1, n\}$, we also have $2f(n - 5)$ linear extensions.

In total, there are $4f(n-4)+4f(n-5)+f(n-6)$ linear extensions, so $f(n) = 4f(n-4)+4f(n-5)+f(n-6)$. From the recursive definition $F_n = F_{n-1} + F_{n-2}$ of the Fibonacci numbers, we also have $F_n = 4F_{n-4} + 4F_{n-5} + F_{n-6}$. Therefore, since $f(n) = F_n$ for $0 \leq n \leq 5$, by induction $f(n) = F_n$ for all positive integers n. □

5. Random Graphs

For real-world problems, it is not always known with a high degree of certainty whether some edge exists or not. For example, in the case of disease spreading, instead of knowing for sure whether two individuals met at a certain point, we usually will only know the probability that they have met based, for example, on their locations. In this case, such a graph would be modeled with probabilities associated to each edge, and would be known as a *random graph*. Staying with the example of disease spread, the problem would still be to determine the most probable sources of the disease, equivalent to finding the orientations that maximize the number of linear extensions in the case of non-random graphs. However, for random graphs, orientations and linear extensions are not even well-defined. In this section we introduce intuitive extensions for each of these concepts to random graphs, and attempt to extend the previous conjectures and theorems to random graphs as well.

It is possible for an edge of a random graph to have an associated probability of 0. If this is the case, that edge can be effectively ignored, simplifying somewhat the heavy calculations. Even more, edges with sufficiently small probabilities could sometimes be assumed to have an effective probability of 0. Firstly, we define an orientation on a random graph.

Definition 7. *Given a random graph with n vertices, an acyclic orientation over the random graph is defined to be an acyclic orientation over the complete graph K_n on n vertices, i.e. a* transitive tournament *on K_n.*

Since there are $n!$ acyclic orientations of the complete graph K_n, there are $n!$ acyclic orientations over any random graph with n vertices. Now, since a random graph can be effectively regarded as the set of all subgraphs of the corresponding complete graph, each with an associated probability of existence, the orientation of the random graph can now be extended to an orientation of each of its subgraphs. This is simple: If an edge exists in a subgraph of the random graph, then that edge is oriented in the same way as it is in the random graph orientation.

With this in mind, we can define the number of linear extensions on a random graph with a given acyclic orientation.

Given a random graph $G(V, \binom{V}{2}, p)$ with n vertices, where $p : \binom{V}{2} \to [0, 1]$ is the existence probability function, and an acyclic orientation O of this graph, there are $2^{\binom{n}{2}}$ possible subgraphs of the random graph, counting the number of subsets of the $\binom{n}{2}$ edges. All such subgraphs have an associated probability equal to the probability of that subgraph occurring under p. That is, given a subgraph $H(V, E)$ of G, the probability p_H that it occurs is equal to:

$$p_H := \prod_{e \in E} p_e \prod_{f \notin E} (1 - p_f),$$

where p_e is the probability associated to each edge. Now, denote by $\varepsilon_O(H)$ the number of linear extensions of the simple subgraph H with acyclic orientation obtained (by restriction) from O.

Definition 8. *The number of linear extensions of the random graph G with acyclic orientation O is:*

$$\sum_{H \in S} p_H \varepsilon_O(H),$$

where S is the set of $2^{\binom{n}{2}}$ subgraphs of the complete graph on n vertices.

Even with computer assistance, to find this value is computationally unviable for all but the simplest graphs. Here I illustrate a simple example to show the kind of computation involved.

Example 3.

Suppose we have a random graph on 3 vertices, labeled a, b, and c. Suppose the orientation on this complete graph is $(a, b), (b, c), (a, c)$. Also suppose the probability associated with (a, b) is $\frac{1}{3}$, the probability associated with (b, c) is $\frac{1}{2}$, and the probability associated with (a, c) is $\frac{2}{3}$. There are $2^{\binom{3}{2}} = 8$ possible subgraphs.

The probability of a complete graph is $\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{9}$. There is only one linear extension in this case.

The probability of the graph containing only (a, b) and (b, c) is $\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{18}$. Again, this has only one linear extension.

The probability of the graph containing only (a, b) and (a, c) is $\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{9}$. This subgraph has two possible linear extensions.

The probability of the graph containing only (b, c) and (a, c) is $\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{2}{9}$. Again, there are two possible linear extensions.

The probability of the graph containing only (a, b) is $\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{18}$. This graph has three possible linear extensions.

The probability of the graph containing only (b, c) is $\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{9}$. This graph also has three possible linear extensions.

The probability of the graph containing only (a, c) is $\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{2}{9}$. This graph also has three possible linear extensions.

The probability of the graph containing no edges is $\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{9}$. This graph has $3! = 6$ linear extensions.

In total, the number of linear extensions associated with this orientation of this probability graph is

$$\left(\frac{1}{9}\right)(1) + \left(\frac{1}{18}\right)(1) + \left(\frac{1}{9}\right)(2) + \left(\frac{2}{9}\right)(2) + \left(\frac{1}{18}\right)(3) + \left(\frac{1}{9}\right)(3) + \left(\frac{2}{9}\right)(3) + \left(\frac{1}{9}\right)(6)$$
$$= \frac{8}{3}.$$

□

This example illustrates an important point in calculating the number of linear extensions of a random graph. Firstly, subgraphs of the complete graph are not necessarily connected, and it may even be the case that some vertices are not connected to any others, but they are still factored in when determining the number of linear extensions of the subgraph. Secondly, the number of linear extensions of a random graph is not necessarily an integer number.

This is only one of the 6 possible orientations of this random graph. I will not go through the rest of the calculation in this paper, but the result is that this orientation, along with the orientation $(b, a), (b, c), (c, a)$, results in the maximum number of linear extensions. Intuition agrees with this result - in both orientations b is not the originator of the disease, which is intuitively most probable because the edges connected to b have the lowest probabilities.

I now present a theorem that expresses the number of linear orientations of a random graph in a different way, possibly reducing the amount of calculation needed. Let G be a random graph on n vertices with some acyclic orientation O, and let $\mathcal{R}$ denote the set of $n!$ total orderings of vertices of the complete graph K_n. For some total ordering $r \in \mathcal{R}$, let $p_{r,O}$ denote the sum of the existence probabilities of all directed subgraphs of G for which r is a valid linear extension, or equivalently, the probability that r is a viable linear extension.

Theorem 10. *The number of linear extensions of G with acyclic orientation O is also equal to:*

$$\sum_{r \in \mathcal{R}} p_{r,O}.$$

Here, instead of summing $2^{\binom{n}{2}}$ terms over all the possible subgraphs, we are only summing $n!$ terms over all total orderings of n vertices, a significant improvement. Also, using this, we are no longer required to have to find the number of linear extensions for each of these subgraphs, a task that is NP-hard.

Proof. This value is equal to $\sum_{r \in \mathcal{R}} E[\text{graphs with } r \text{ as a linear extension}]$ by linearity of expectation, and this is equal to $E[\text{number of linear extensions}]$ again by linearity of expectation, which is the original definition of the number of linear extensions of a random graph. □

Now, with two formulations for the number of linear extensions for a random graph, we can now extend the Max-Cut Problem to random graphs. This is known as the *Weighted Max-Cut Problem.*

Problem 2. *Given a random graph G, determine a partition of the vertices of G into two classes such that the sum of the probabilities of each edge with one vertex in each class is maximized.*

For the case of the random graph on 3 vertices as defined in Example 2, the solution to the Weighted Max-Cut Problem can easily be found to be the partition $\{a, b\}$ and $\{c\}$.

To work through some more manageable cases, define a *random bipartite graph* to be a random graph whose vertices can be partitioned into two subsets such that the probability of an edge between any two members of the same subset is 0.

Theorem 11. *Given a random bipartite graph and a solution to the Weighted Max-Cut Problem for that graph, there exists an orientation of the graph that is both bipartite with respect to the optimal partition, and also maximizes the number of linear extensions.*

Proof. Firstly, it is clear that the solution to the Weighted Max-Cut Problem for a random bipartite graph is exactly the bipartite partition. Now, I claim that, in a random bipartite partition, the orientations that maximizes the number of linear extensions are exactly the bipartite orientations (since all edges between vertices in the same partition have probability 0, these edges are effectively ignored with regards to the orientation.) Consider the first definition of the number of linear extensions of a random graph.

$$\sum_{H \in S} p_H \varepsilon_O(H),$$

where the variables are defined the same as in Definition 7. In each term, the value of p_H is constant with respect to the orientation. Also, removing edges from a bipartite graph still results in a bipartite graph, and by Theorem 1, it is known that the orientations that maximize the number of linear extensions in a bipartite graph are the bipartite orientations. Therefore, if O is a bipartite orientation for the random graph, then it is also a bipartite orientation for all subgraphs of the random graph, and it therefore maximises $\varepsilon_O(H)$ for all subgraphs H. Therefore, since each individual term is maximized with a bipartite orientation, then the entire sum is maximized with a bipartite orientation, and so the orientations that maximize the number of linear extensions in a random bipartite graph are exactly the bipartite orientations. □

In essence, this extends Conjecture 1 to random bipartite graphs; again, however, this is not true in general.

6. Acknowledgements

I would like to thank my mentor, Benjamin Iriarte, for his continued support and special insights into my research. I would also like to thank Dr. Tanya Khovanova for her suggestions on writing this paper, and the MIT PRIMES program, Professors Slava Gerovitch and Pavel Entingof in particular, for providing the means and the opportunity to undergo this project.

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