Discrete and Continuous Dynamical Systems: Applications and Examples

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Overview of dynamical systems

What is a dynamical system?

Two flavors:

- Discrete (Iterative Maps)

- Continuous (Differential Equations)

Iterative maps

Definition (Iterative map)

A *(one-dimensional) iterative map* is a sequence $\{x_n\}$ with $x_{n+1} = f(x_n)$ for some function $f : \mathbb{R} \to \mathbb{R}$.

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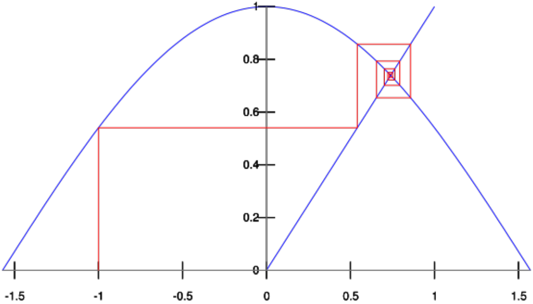
- Periodic points (can be reduced to fixed points)
- Stability of fixed points
- By approximating f with a linear function, we get that a fixed point x^* is stable whenever $|f'(x^*)| < 1$.

Getting a picture: “cobwebbing”



A famous example: the logistic map

We consider

$$x_{n+1} = rx_n(1 - x_n)$$

on the interval $[0, 1]$.

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- If $3 < r \leq 1 + \sqrt{6}$, no stable fixed points, but $\frac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r}$ is stable 2-cycle.

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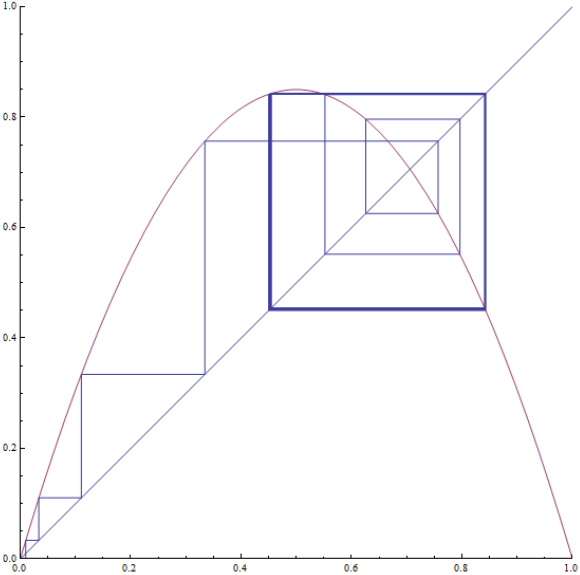
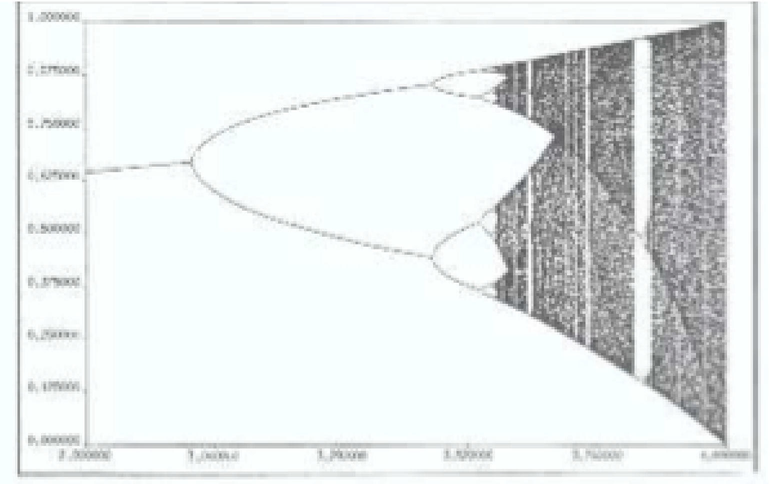
- 2-cycle becomes 4-cycle, then 8-cycle, and so on.

The case of a stable 2-cycle



The orbit diagram

We can plot the points in stable cycles with respect to r:



The first Feigenbaum constant

Let r_n be where stable 2^n cycle begins.
The distance between r_n's converges roughly geometrically, up to r_∞.

Definition (δ)

The *first Feigenbaum constant* is defined as

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} \approx 4.669\ldots$$

The first Feigenbaum constant: not just for one map?

Yeah, but why do we care about δ?

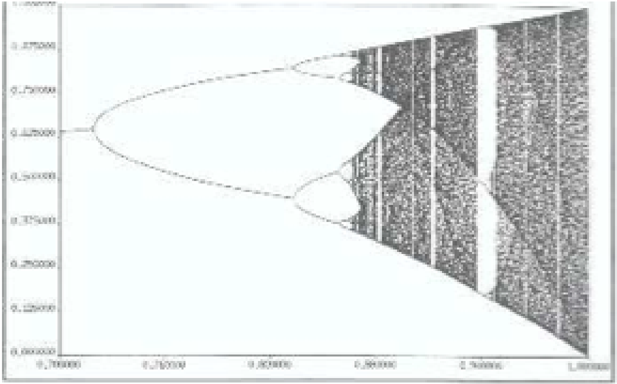
Consider the *sine map*

$$x_{n+1} = r \sin \pi x_n.$$

Guess what its orbit diagram looks like?

Sine map orbit diagram

No, I didn't accidentally repeat the previous image...



...it looks exactly the same!
Not only that, if you try to calculate δ here, you'll get the same number!

Univerality of δ

Theorem 1 (Universality of δ)

If

$$D_{sch}f(x) = \left(\frac{f''}{f'}\right)'(x) - \frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 < 0$$

in the bounded interval and f experiences period-doubling, then letting $\{r_n\}$ be defined for this new map,

$$\lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = \delta.$$

Essentially, δ is a "*universal constant*!"

Now, the continuous case ...

- Continuous dynamical systems involve analyzing differential equations.
- They describe systems that change over time.

Oscillating chemical reactions

- Chemical reactions: governed by differential equations involving concentrations of the reactants and products.
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- Multi-step reactions can exhibit complicated dynamical behaviors.
- Belousov's discovery in 1950's exhibits a periodical behavior.

Continuous dynamical systems and oscillating chemical reactions

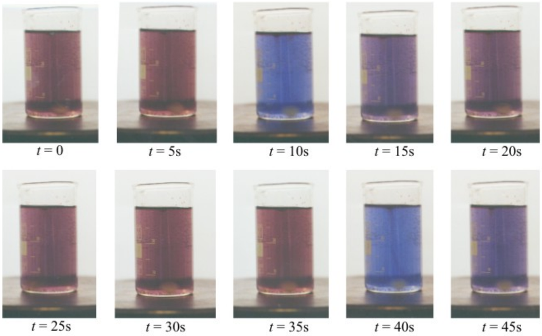

Figure: Periodic behavior of an oscillating chemical reaction.

Continuous dynamical systems: one–dimensional case

- $\dot{x} = f(x)$
- The continuous time dynamics $\dot{x}$ of a system is governed by its current state x.

Continuous dynamical systems: one–dimensional case

- Example: $\dot{x} = r + x^2$, where r is a parameter.

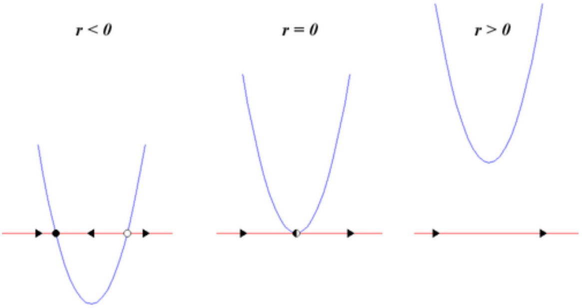

Figure: The phase portrait of the system $\dot{x} = r + x^2$.

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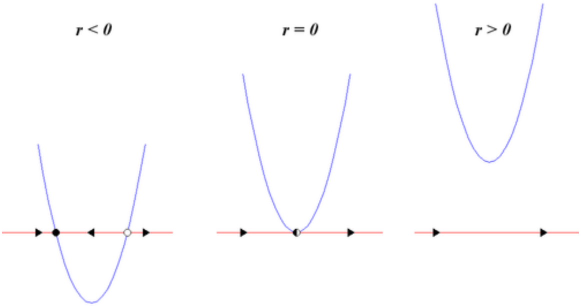

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- Flow and vector fields
- Stable and unstable fixed points ($\dot{x} = 0$)

Continuous dynamical systems: bifurcations

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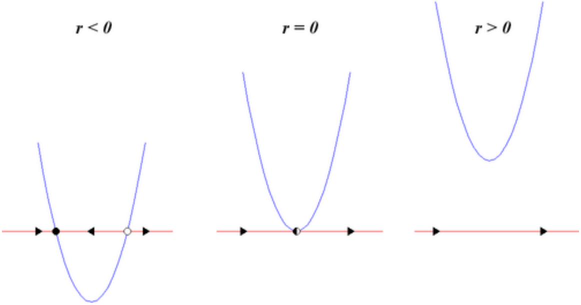

Figure: The phase portrait of the system $\dot{x} = r + x^2$.

- Bifurcation: a qualitative change in the vector field.

Continuous dynamical systems: two–dimensional case

- $\dot{x} = f(x, y)$
  $\dot{y} = g(x, y)$.

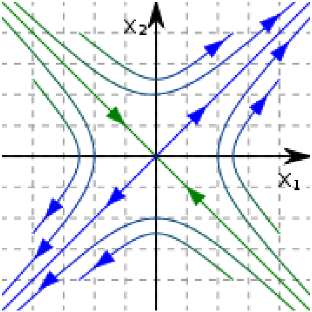

Figure: A two-dimensional vector field.

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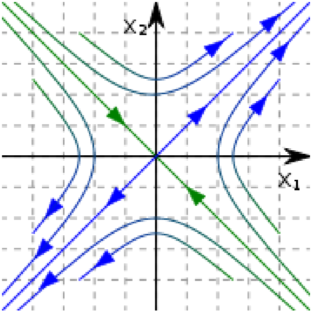

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- Vector fields: represented as arrows on the plane (phase portrait).

Linearization near a fixed point

- Linearized systems: near a fixed point (x^*, y^*),

$$u = x - x^*, v = y - y^*,$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- The matrix

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*,y^*)}$$

  is called the Jacobian matrix at the fixed point (x^*, y^*).

Linearization near a fixed point

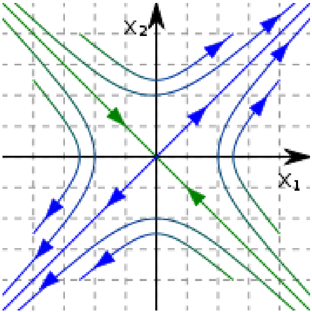

Figure: A two-dimensional vector field.

- The eigenvectors and eigenvalues (λ) of A determine the eigendirections near (x^*, y^*).
- Behavior of the flow near a fixed point is governed by the stable manifolds ($\lambda < 0$) and unstable manifolds ($\lambda > 0$).

Back to oscillating chemical reactions...

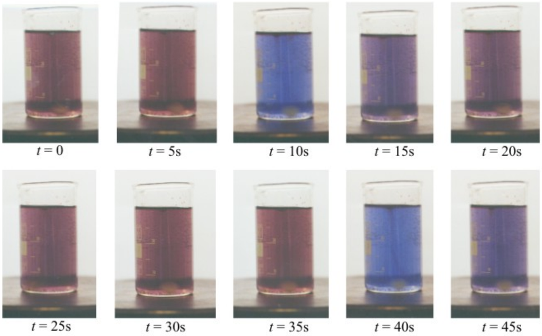

Figure: Periodic behavior of an oscillating chemical reaction.

The BZ (Belousov-Zhabotinsky) reaction

Main reaction steps:

$$MA + I_2 \rightarrow IMA + I^- + H^+; \frac{d[I_2]}{t} = -\frac{k_{1a}[MA][I_2]}{k_{1b} + [I_2]} \tag{1}$$

$$ClO_2 + I^- \rightarrow ClO_2^- + \frac{1}{2}I_2; \frac{d[ClO_2]}{t} = -k_2\frac{[ClO_2]}{[I^-]} \tag{2}$$

$$ClO_2^- + 4I^- + 4H^+ \rightarrow Cl^- + 2I_2 + 2H_2O;$$

$$\frac{d[ClO_2^-]}{dt} = -k_{3a}[ClO_2^-][I^-][H^+] - k_{3b}[ClO_2^-][I_2]\frac{[I^-]}{u + [I^-]^2} \tag{3}$$

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$\Longrightarrow$ Very complicated.

Simplified model of the BZ reaction

- $\dot{x} = a - x - \frac{4xy}{1+x^2}$,
  $\dot{y} = bx\left(1 - \frac{y}{1+x^2}\right)$.
  Here, x and y are dimensionless concentrations of I^- and ClO_2^-.

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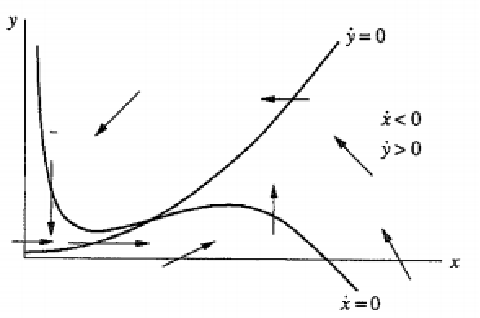

Figure: The phase portrait of the simplied model of the BZ reaction.

- Fixed point where the nullclines intersect tangentially

Analysis of the dynamical system

- Behavior of vector fields near a fixed point in a linearized system is determined by the determinant Δ and the trace τ of the Jacobian matrix.

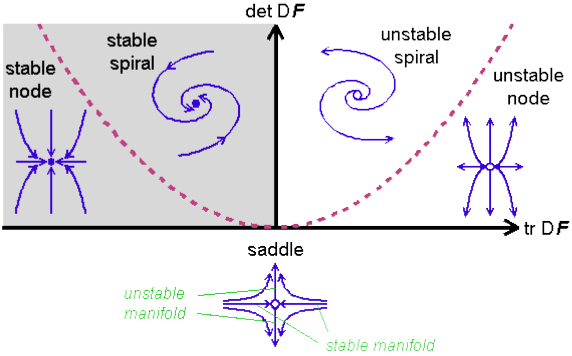

Figure: Classification of fixed points.

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- The determinant and trace are given by

$$\Delta = \frac{5bx^*}{1+(x^*)^2} > 0, \tau = \frac{3(x^*)^2 - 5 - bx^*}{1+(x^*)^2}.$$

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- $\tau > 0$ if $b < b_c = 3a/5 - 25/a$.
- A bifurcation occurs at $b = b_c$ (the stability of the fixed point changes).

Applying the Poincaré-Bendixson Theorem

Theorem 2 (Poincaré-Bendixson Theorem)

Suppose that:

1. *R is a closed, bounded subset of the plane;*
2. *$\dot{x} = f(x)$ is a continuously differentiable vector field on an open set containing R;*
3. *R does not contain any fixed points; and*
4. *There exists a trajectory C that is "confined" in R, in the sense that it starts in R and stays in R for all future time.*

Then R contains a closed orbit.

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Conclusion

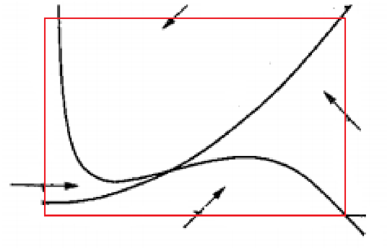

Figure: A trapping box in the BZ reaction system.

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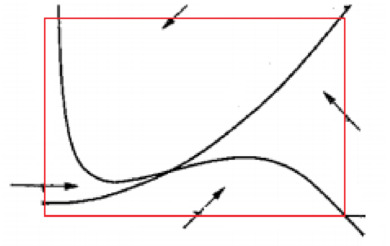

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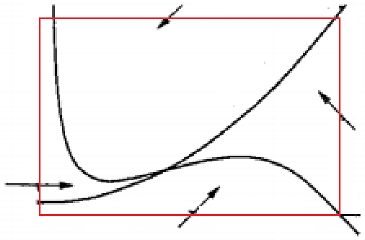

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- Chemically, this explains why the BZ reaction shows a periodic behavior.

Conclusion

- Changing the parameters $a, b > 0$, which depend on the rate constants and concentrations of slow reactants, results in a supercritical Hopf bifurcation.
- Change in stability.
- Formation of a stable limit cycle.

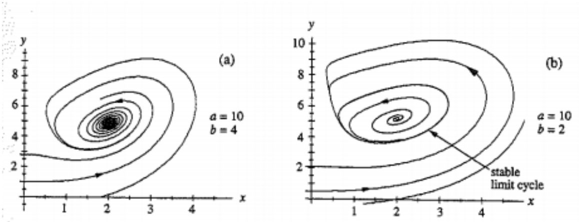

Figure: Supercritical Hopf bifurcation in the BZ reaction system.

Bibliography

S. H. Strogatz,
Nonlinear Dynamics and Chaos,
Perseus Books Publishing, Cambridge, 1994.

Acknowledgements

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