On the Existence of Srg(99, 14, 1, 2)

By Andrew He, Suzy Lou, and Max Murin

Mentor: Dr. Peter Csikvari

Fourth Annual PRIMES Conference

Acknowledgements

Thanks to:

More definitions

Our project: What about srg(99, 14, 1, 2)?

Progress on Idea 3

- Triangles are classified by distance from a central triangle V
- γ: number of triangles with exactly vertices of distance 1 from V, with integer values from 0 to 12

Progress on idea 1

With this labeling, structural features of srg(99, 14, 1, 2) are tied to decompositions of srg(14, 12, 10, 12) into disjoint polygons (Note: srg(14, 12, 10, 12) is simply the complete graph on 14 vertices, minus 7 disjoint edges

More on idea 1

Progress on Idea 4

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Definitions:

Strongly Regular Graph:

An srg(v, k, a, b) is a graph with v vertices, each of degree k. Every pair of adjacent vertices have a common neighbors. Every pair of non-adjacent vertices have b common neighbors.

Example

A quadrilateral: srg(4, 2, 0, 2)! Each vertex has degree 2, adjacent vertices share no neighbors, non-adjacent vertices share 2 neighbors

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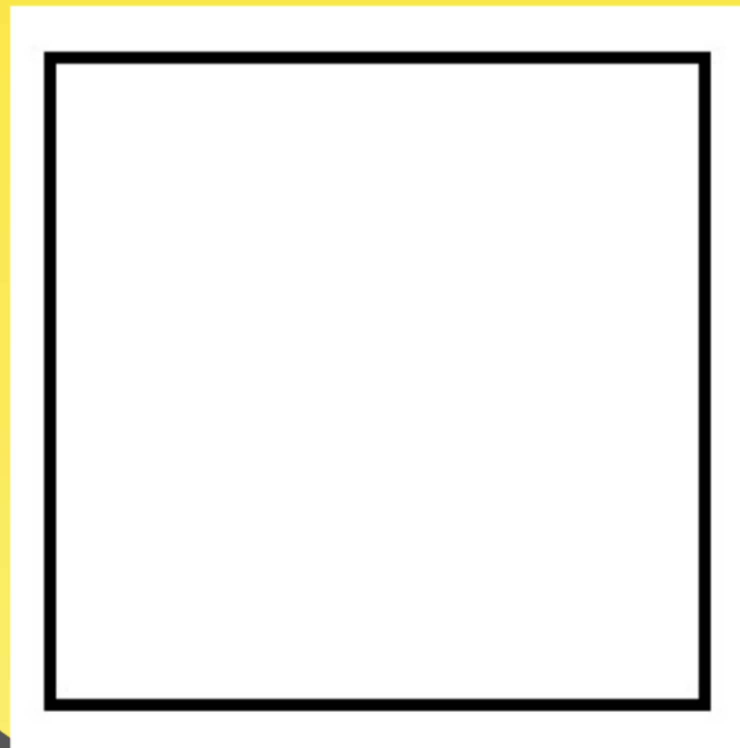
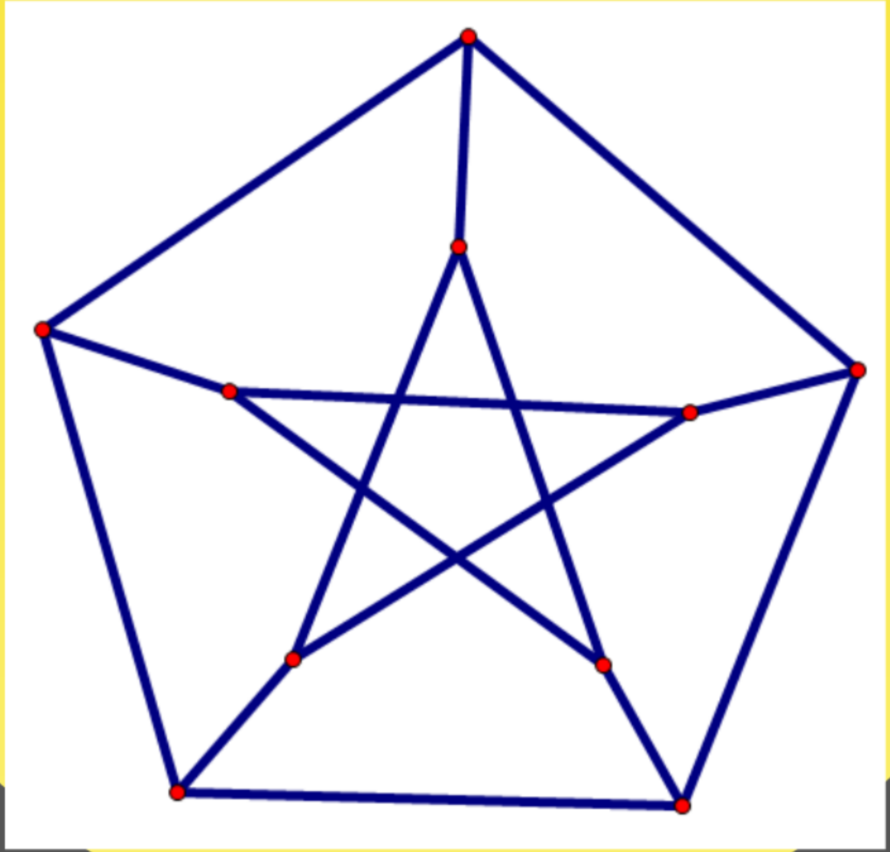


Example: Complete Graphs

- Each vertex has same degree
- Each pair of vertices (hence each pair of adjacent vertices) shares same number of vertices in common
- There are no non-adjacent vertices, so the last condition is trivially met
- (Though this triviality leads some to exclude complete graphs from strongly regular graphs)

Example

The Petersen graph is srg(10, 3, 0, 1).



Another Example

Srg(16, 6, 2, 2)
Use a clever labeling
Take the sixteen points of $(\mathbb{Z}/4\mathbb{Z})^2$
Let (a, b) be connected to (c, d) iff $a = c$ or $b = d$. Then, every point has six neighbors, so $k = 6$.
(Verification of other conditions is left as an exercise :))

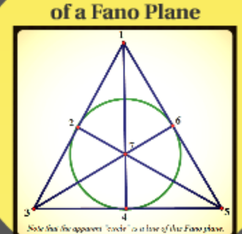
More definitions

Fano plane:
A set of 7 "lines" that each contain three points that come from a set of seven points. Any two lines intersect in exactly one point.

Motivation: As seen before, a clever labeling can be helpful

A Fano-Plane may provide a useful labeling

Traditional Picture of a Fano Plane



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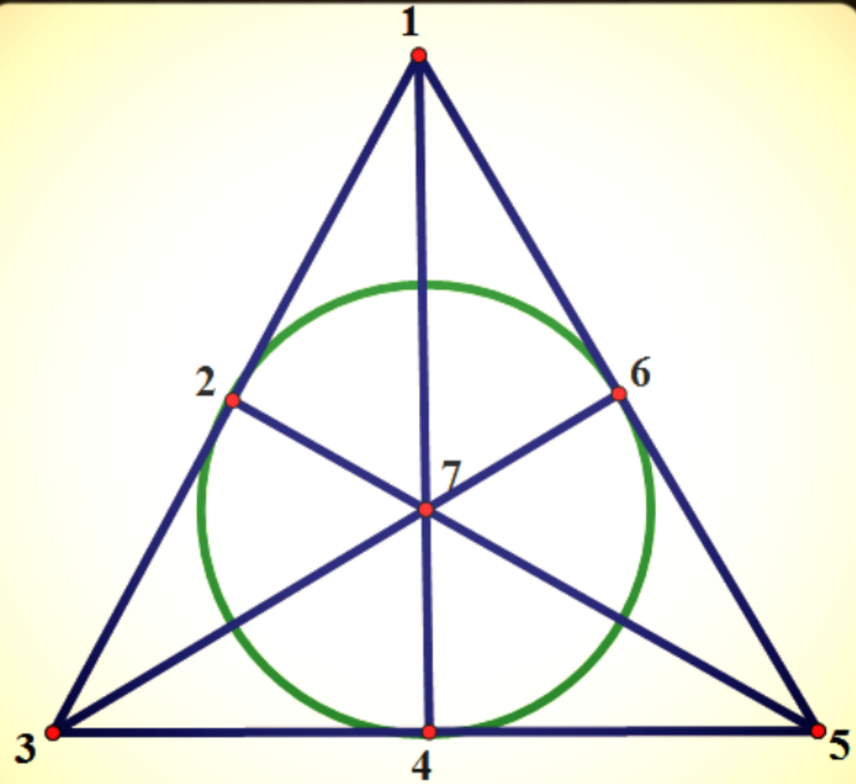

Note that the apparent "circle" is a line of this Fano plane.

Our project: What about srg(99, 14, 1, 2)?

Does it exist?

Thoughts:
Two unconnected vertices define a quadrilateral

More Thoughts

Because of the third parameter (1), every edge is a part of exactly one triangle

Implication: the fourteen neighbors of a vertex are grouped into seven triangles

(i.e. each vertex is hinged on seven triangles)

Possible labellings

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Possible labelings

1. Call a central vertex V and its fourteen neighbors 1, 2, ..., 14. Let vertex 1 be connected to 2, 3 to 4, etc.
2. Given a set of seven elements, there exist two disjoint sets of 15 Fano-planes with points from that set. Let a vertex and its fourteen neighbors be labeled with 15 Fano-planes in a set.
3. Use the fact that the graph, if it exists, has a triangle decomposition. Examine the triangles.
4. Examine the largest independent set

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- The most obvious polygonal split of the graph is a certain quadrilateral split
- However, this quadrilateral split did not translate into a viable structure in the graph

Progress on Idea 2

Prezi

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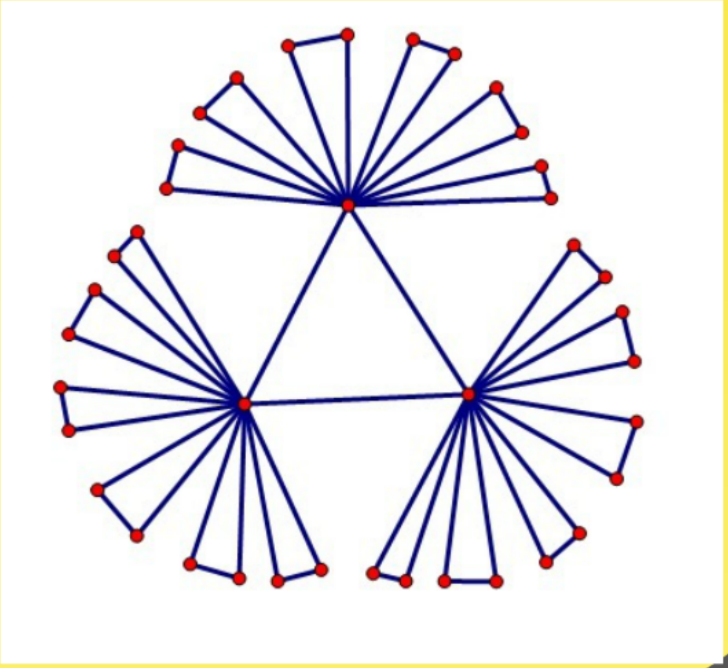
Progress on Idea 3

Idea is based on the fact that every vertex is hinged upon seven triangles; examine structure of triangles



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- Lemma: $\gamma \neq 11$
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Progress on Idea 4

- Examine largest independent set *I*
- Upper bound for size of *I*: 22
- Conjecture: *I* has size 22

Progress on Idea 4

- Theorem: If *I* indeed has the maximum theoretical size (22) each vertex not in *I* is connected to exactly 4 vertices in *I*.

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Proof of claim

- Let I be of size n.
- Set, S, of all vertices that are a neighbor of at least one member of I has size $99-n$
- Number of edges between I and S: $14n$
- Let s=vertex in S. $F(s)$=number of edges between S and I adjacent to s.

$$\sum_s F(s) = 14n$$

- 2 nonadjacent vertices share 2 neighbors, so

$$\sum_s \binom{F(s)}{2} = 2\binom{n}{2} = n^2 - n$$

- Then $\sum_s F(s)^2 = 2n^2 + 12n$

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Proof of Claim (cont.)

- By the RMS-AM inequality, this turns into

$$-n^2 - 5n + 594 \geq 0$$

- $27 \leq n \leq 22$

- Equality holds when all elements are equal
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- Consequences: vertices of I can be placed into "blocks" of size 4.
- For any two vertices in I, there are two blocks that contain both of them.
- There are not many ways to arrange blocks in the specified way
- These blocks may give lots of help with structure

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Acknowledgements

Thanks to:

- Dr. Peter Csikvari, our mentor (who also suggested our problem)
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