G-Parking Functions and Monomial Ideals

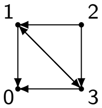
Brice Huang

Mentored by Wuttisak Trongsiriwat
Fourth Annual MIT PRIMES Conference

May 17, 2014

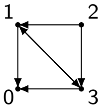
Definitions

- Directed graph (digraph): collection of vertices and oriented edges between pairs of vertices

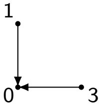


Definitions

- Directed graph (digraph): collection of vertices and oriented edges between pairs of vertices

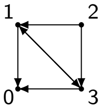


- Subtree of digraph: subgraph in which each vertex has a unique path to a vertex known as the root

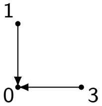


Definitions

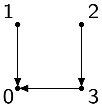
- Directed graph (digraph): collection of vertices and oriented edges between pairs of vertices



- Subtree of digraph: subgraph in which each vertex has a unique path to a vertex known as the root

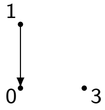


- Spanning tree of digraph: a subtree containing all vertices



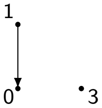
Definitions

- Subforest of digraph: the union of one or more subtrees on disjoint sets of vertices

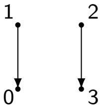


Definitions

- Subforest of digraph: the union of one or more subtrees on disjoint sets of vertices



- Spanning forest of digraph: a subforest containing all vertices



Parking Functions

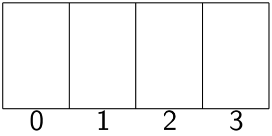
- A *classical parking function* is an n-tuple of nonnegative integers $(b_1, \ldots, b_n)$ that, when sorted in decreasing order, is termwise less than $(n, n-1, \ldots, 1)$

Parking Functions

- A *classical parking function* is an n-tuple of nonnegative integers $(b_1, \ldots, b_n)$ that, when sorted in decreasing order, is termwise less than $(n, n-1, \ldots, 1)$
- Analogy: n drivers on a one-way road with parking spots $0, 1, \ldots, n-1$

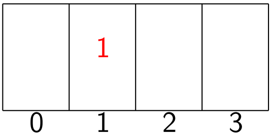
Parking Functions

- A *classical parking function* is an n-tuple of nonnegative integers $(b_1, \ldots, b_n)$ that, when sorted in decreasing order, is termwise less than $(n, n-1, \ldots, 1)$
- Analogy: n drivers on a one-way road with parking spots $0, 1, \ldots, n-1$
- Example: (1,0,3,0)



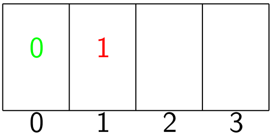
Parking Functions

- A *classical parking function* is an n-tuple of nonnegative integers $(b_1, \ldots, b_n)$ that, when sorted in decreasing order, is termwise less than $(n, n-1, \ldots, 1)$
- Analogy: n drivers on a one-way road with parking spots $0, 1, \ldots, n-1$
- Example: (1,0,3,0)



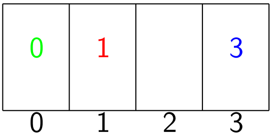
Parking Functions

- A *classical parking function* is an n-tuple of nonnegative integers $(b_1, \ldots, b_n)$ that, when sorted in decreasing order, is termwise less than $(n, n-1, \ldots, 1)$
- Analogy: n drivers on a one-way road with parking spots $0, 1, \ldots, n-1$
- Example: (1,0,3,0)



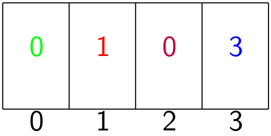
Parking Functions

- A *classical parking function* is an n-tuple of nonnegative integers $(b_1, \ldots, b_n)$ that, when sorted in decreasing order, is termwise less than $(n, n-1, \ldots, 1)$
- Analogy: n drivers on a one-way road with parking spots $0, 1, \ldots, n-1$
- Example: $(1,0,3,0)$



Parking Functions

- A *classical parking function* is an n-tuple of nonnegative integers $(b_1, \ldots, b_n)$ that, when sorted in decreasing order, is termwise less than $(n, n-1, \ldots, 1)$
- Analogy: n drivers on a one-way road with parking spots $0, 1, \ldots, n-1$
- Example: (1,0,3,0)



Parking Functions

- There are $(n+1)^{n-1}$ classical parking functions of size n

Parking Functions

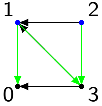
- There are $(n+1)^{n-1}$ classical parking functions of size n

Theorem (Cayley)

The complete graph K_{n+1} has $(n+1)^{n-1}$ spanning trees.

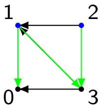
G-Parking Functions

- G is a digraph on vertices $\{0, 1, \ldots, n\}$
- For a nonempty subset $I \subseteq \{1, \ldots, n\}$, and vertex $i \in I$, let $d_I(i)$ denote the number of edges from i to vertices outside I



G-Parking Functions

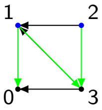
- G is a digraph on vertices $\{0, 1, \ldots, n\}$
- For a nonempty subset $I \subseteq \{1, \ldots, n\}$, and vertex $i \in I$, let $d_I(i)$ denote the number of edges from i to vertices outside I



- A G-parking function is an n-tuple $(b_1, \ldots, b_n)$ such that for any nonempty subset $I \subseteq \{1, 2, \ldots, n\}$, there exists $i \in I$ such that $b_i < d_I(i)$

G-Parking Functions

- G is a digraph on vertices $\{0, 1, \ldots, n\}$
- For a nonempty subset $I \subseteq \{1, \ldots, n\}$, and vertex $i \in I$, let $d_I(i)$ denote the number of edges from i to vertices outside I



- A G-parking function is an n-tuple $(b_1, \ldots, b_n)$ such that for any nonempty subset $I \subseteq \{1, 2, \ldots, n\}$, there exists $i \in I$ such that $b_i < d_I(i)$
- Example: $(0, 1, 1)$ is a G-parking function, where G is the graph above

G-Parking Functions

- Theorem
  The number of G-parking functions equals the number of spanning trees of G rooted at 0.

G-Parking Functions

- Theorem
  The number of G-parking functions equals the number of spanning trees of G rooted at 0.
  - Classical parking functions are the special case $G = K_{n+1}$

G-Parking Functions

- Theorem
  The number of G-parking functions equals the number of spanning trees of G rooted at 0.
  - Classical parking functions are the special case $G = K_{n+1}$
  - Chebikin and Pylyavskyy constructed an explicit bijection

Chebikin-Pylyavskyy Bijection

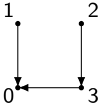
- For every subtree T of G rooted at 0, assign an order $\pi(T)$ to T's vertices. Let $i <_{\pi(T)} j$ denote i being smaller than j in this order

Chebikin-Pylyavskyy Bijection

- For every subtree T of G rooted at 0, assign an order $\pi(T)$ to T's vertices. Let $i <_{\pi(T)} j$ denote i being smaller than j in this order
- An choice of orders $\Pi(G)$ is a *proper set of tree orders* if for each subtree T rooted at 0:
  - if an edge $(i, j) \in T$, then $i >_{\pi(T)} j$
  - if t is a subtree of T, then the orders $\pi(t)$ and $\pi(T)$ are consistent

Chebikin-Pylyavskyy Bijection

- For every subtree T of G rooted at 0, assign an order $\pi(T)$ to T's vertices. Let $i <_{\pi(T)} j$ denote i being smaller than j in this order
- An choice of orders $\Pi(G)$ is a *proper set of tree orders* if for each subtree T rooted at 0:
  - if an edge $(i, j) \in T$, then $i >_{\pi(T)} j$
  - if t is a subtree of T, then the orders $\pi(t)$ and $\pi(T)$ are consistent
- Example: Breadth-first search order



$0 <_{\pi(T)} 1 <_{\pi(T)} 3 <_{\pi(T)} 2$

Chebikin-Pylyavskyy Bijection

- Fix a proper set of tree orders $\Pi(G)$
- For each spanning tree T, let $e(T, i)$ be the edge out of i in T
- Given a subtree T and order $\pi(T)$, for each vertex i, order the edges from i to T such that $(i, j_1) <_{\pi(T)} (i, j_2)$ if $j_1 <_{\pi(T)} j_2$

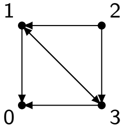
Chebikin-Pylyavskyy Bijection

- Fix a proper set of tree orders $\Pi(G)$
- For each spanning tree T, let $e(T, i)$ be the edge out of i in T
- Given a subtree T and order $\pi(T)$, for each vertex i, order the edges from i to T such that $(i, j_1) <_{\pi(T)} (i, j_2)$ if $j_1 <_{\pi(T)} j_2$

Theorem (Chebikin, Pylyavskyy)

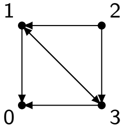
Map each spanning tree T to $(b_1, \ldots, b_n)$, where b_i is the number of edges e from i such that $e <_{\pi(T)} e(T, i)$. This mapping is a bijection between G's spanning trees rooted at 0 and G-parking functions.

Chebikin-Pylyavskyy Bijection - An Example

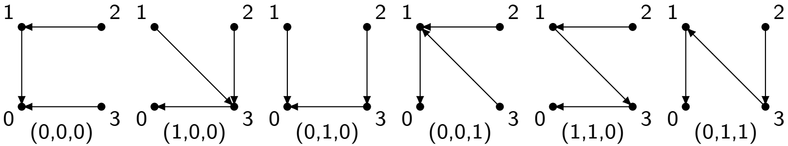


- G-parking functions: (0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1)

Chebikin-Pylyavskyy Bijection - An Example



- *G*-parking functions: (0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1)
- Spanning trees:



Monomial Ideals

- $\mathbb{K}[x_1, \ldots, x_n]$ is the polynomial ring in variables $x_1, \ldots, x_n$ on a fixed field $\mathbb{K}$ of characteristic 0

Monomial Ideals

- $\mathbb{K}[x_1, \ldots, x_n]$ is the polynomial ring in variables $x_1, \ldots, x_n$ on a fixed field $\mathbb{K}$ of characteristic 0
- For each nonempty subset $I \subseteq \{1, \ldots, n\}$, define

$$m_I = \prod_{i \in I} x_i^{d_I(i)}$$

and let the ideal $\mathcal{I}_G = \langle m_I \rangle$ as I ranges over all nonempty subsets of $\{1, \ldots, n\}$

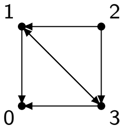
Monomial Ideals

- $\mathbb{K}[x_1, \ldots, x_n]$ is the polynomial ring in variables $x_1, \ldots, x_n$ on a fixed field $\mathbb{K}$ of characteristic 0
- For each nonempty subset $I \subseteq \{1, \ldots, n\}$, define

$$m_I = \prod_{i \in I} x_i^{d_I(i)}$$

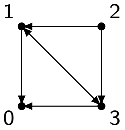
  and let the ideal $\mathcal{I}_G = \langle m_I \rangle$ as I ranges over all nonempty subsets of $\{1, \ldots, n\}$
- $(b_1, \ldots, b_n)$ is a G-parking function if and only if $x_1^{b_1} \cdots x_n^{b_n}$ does not vanish in $\mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_G$

Monomial Ideals - An Example



- $\mathcal{I}_G = \langle x_1^2, x_2^2, x_3^2, x_1^2 x_2, x_1 x_3, x_2 x_3^2, x_1 x_2^0 x_3 \rangle$

Monomial Ideals - An Example



- $\mathcal{I}_G = \langle x_1^2, x_2^2, x_3^2, x_1^2 x_2, x_1 x_3, x_2 x_3^2, x_1 x_2^0 x_3 \rangle$
- Non-vanishing monomials: $1, x_1, x_2, x_3, x_1 x_2, x_2 x_3$
- G-parking functions: (0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1)

Almost-G-Parking Functions

- For each nonempty subset $I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, n\}$, define
$$\hat{m}_I = x_{i_1} \prod_{i \in I} x_i^{d_I(i)}$$
and let the ideal $\hat{\mathcal{I}}_G = \langle \hat{m}_I \rangle$ as I ranges over all nonempty subsets of $\{1, \ldots, n\}$
- $(b_1, \ldots, b_n)$ is an *almost-G-parking function* if $x_1^{b_1} \cdots x_n^{b_n}$ does not vanish in $\mathbb{K}[x_1, \ldots, x_n]/\hat{\mathcal{I}}_G$

Almost-G-Parking Functions

- For each nonempty subset $I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, n\}$, define
$$\hat{m}_I = x_{i_1} \prod_{i \in I} x_i^{d_I(i)}$$
and let the ideal $\hat{\mathcal{I}}_G = \langle \hat{m}_I \rangle$ as I ranges over all nonempty subsets of $\{1, \ldots, n\}$
- $(b_1, \ldots, b_n)$ is an *almost-G-parking function* if $x_1^{b_1} \cdots x_n^{b_n}$ does not vanish in $\mathbb{K}[x_1, \ldots, x_n]/\hat{\mathcal{I}}_G$

- Theorem (Postnikov, Shapiro, Shapiro)

  When $G = K_{n+1}$, the number of almost-G-parking functions equals the number of (undirected) spanning forests of G.

Almost-G-Parking Functions and Spanning Forests

- We explicity construct a bijection between almost-G-parking functions and spanning forests of G whose connected components are rooted at their smallest vertices

Almost-G-Parking Functions and Spanning Forests

- We explicity construct a bijection between almost-G-parking functions and spanning forests of G whose connected components are rooted at their smallest vertices
- For every subtree T of G rooted at its numerically smallest vertex, assign an order $\hat{\pi}(T)$. Let $i <_{\hat{\pi}(T)} j$ denote i being smaller than j in this order

Almost-G-Parking Functions and Spanning Forests

- We explicity construct a bijection between almost-G-parking functions and spanning forests of G whose connected components are rooted at their smallest vertices
- For every subtree T of G rooted at its numerically smallest vertex, assign an order $\hat{\pi}(T)$. Let $i <_{\hat{\pi}(T)} j$ denote i being smaller than j in this order
- A choice of orders $\hat{\Pi}(G)$ is a *super-proper set of tree orders* if for each subtree T rooted at its numerically smallest vertex:
  - if an edge $(i, j) \in T$, then $i >_{\hat{\pi}(T)} j$
  - if t is a subtree of T with the same root, then the orders $\hat{\pi}(t)$ and $\hat{\pi}(T)$ are consistent

Almost-G-Parking Functions and Spanning Forests

- We explicity construct a bijection between almost-G-parking functions and spanning forests of G whose connected components are rooted at their smallest vertices
- For every subtree T of G rooted at its numerically smallest vertex, assign an order $\hat{\pi}(T)$. Let $i <_{\hat{\pi}(T)} j$ denote i being smaller than j in this order
- A choice of orders $\hat{\Pi}(G)$ is a *super-proper set of tree orders* if for each subtree T rooted at its numerically smallest vertex:
  - if an edge $(i, j) \in T$, then $i >_{\hat{\pi}(T)} j$
  - if t is a subtree of T with the same root, then the orders $\hat{\pi}(t)$ and $\hat{\pi}(T)$ are consistent
- Example: Breadth-first search order

Almost-G-Parking Functions and Spanning Forests

- Given a super-proper set of tree orders $\hat{\Pi}(G)$, for every spanning forest F of G whose connected components are rooted at their numerically smallest vertices, assign an order $\pi(F)$ such that $i <_{\pi(F)} j$ if:
  - the root of i's connected component is smaller than the root of j's connected component, or
  - i and j are in the same connected component T, and $i <_{\hat{\pi}(T)} j$

Almost-G-Parking Functions and Spanning Forests

- Fix a super-proper set of tree orders $\hat{\Pi}(G)$
- Let $e(F, i)$ be the edge out of i in the spanning forest F, if it exists
- Given a subforest F and order $\pi(F)$, for each vertex i, order the edges from i to F such that $(i, j_1) <_{\pi(F)} (i, j_2)$ if $j_1 <_{\pi(F)} j_2$

Almost-G-Parking Functions and Spanning Forests

- Fix a super-proper set of tree orders $\hat{\Pi}(G)$
- Let $e(F, i)$ be the edge out of i in the spanning forest F, if it exists
- Given a subforest F and order $\pi(F)$, for each vertex i, order the edges from i to F such that $(i, j_1) <_{\pi(F)} (i, j_2)$ if $j_1 <_{\pi(F)} j_2$
- Map each spanning forest F whose connected components are rooted at their numerically smallest vertices to $(b_1, \ldots, b_n)$, where b_i is:
  - the number of edges from i to vertices smaller than i in $\pi(F)$, if i is the root of its connected component
  - the number of edges e from i such that $e <_{\pi(F)} e(F, i)$, otherwise

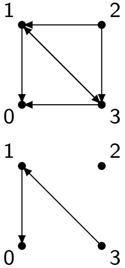
Almost-G-Parking Functions and Spanning Forests

- Fix a super-proper set of tree orders $\hat{\Pi}(G)$
- Let $e(F, i)$ be the edge out of i in the spanning forest F, if it exists
- Given a subforest F and order $\pi(F)$, for each vertex i, order the edges from i to F such that $(i, j_1) <_{\pi(F)} (i, j_2)$ if $j_1 <_{\pi(F)} j_2$
- Map each spanning forest F whose connected components are rooted at their numerically smallest vertices to $(b_1, \ldots, b_n)$, where b_i is:
  - the number of edges from i to vertices smaller than i in $\pi(F)$, if i is the root of its connected component
  - the number of edges e from i such that $e <_{\pi(F)} e(F, i)$, otherwise

Theorem

This mapping is a bijection between almost-G-parking functions and G's spanning forests whose connected components are rooted at their numerically smallest vertices.

An Example



This corresponds to the almost-G-parking function (0,2,1)

Modified Monomial Ideals

For each nonempty $I \subseteq \{1, \dots, n\}$, choose any $k_I \in I$ and let

$$\hat{m}'_I = x_{k_I} \prod_{i \in I} x_i^{d_I(i)}$$

Let $\hat{\mathcal{I}}'_G = \langle \hat{m}'_I \rangle$ as I ranges over all nonempty subsets of $\{1, \dots, n\}$, and let $\hat{\mathcal{A}}'_G = \mathbb{K}[x_1, \dots, x_n]/\hat{\mathcal{I}}'_G$

Theorem

If $G = K_{n+1}$, then $\dim \hat{\mathcal{A}}'_G$ is independent of the choices of k_I.

Future Directions

Conjecture

$\dim \hat{\mathcal{A}}'_G$ *is independent of the choices of* k_I *for all choices of* k_I *that preserve the monotonicity of the ideal* $\hat{\mathcal{I}}'_G$ *(i.e. if* $I \subset J$*, then for any* $i \in I$*,* $\deg_{x_i} \hat{m}'_I \geq \deg_{x_i} \hat{m}'_J$*).*

It would also be interesting to find a combinatorial interpretation of ideals in which the m_I are modified by multiplication by more than one variable

Acknowledgements

Many thanks to:

- My family, for always supporting me
- Wuttisak Trongsiriwat and Professor Alexander Postnikov, for their patience and guidance
- Dr. Slava Gerovitch, Dr. Tanya Khovanova, and the MIT-PRIMES staff, for giving me this opportunity