On the Extremal Functions of Multi-dimensional Matrices

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Fourth Annual MIT PRIMES Conference
May 17th, 2014

0-1 matrix A contains 0-1 matrix B

0	1	1	1
1	0	1	0
1	1	0	1
0	0	0	1
1	0	1	1

A

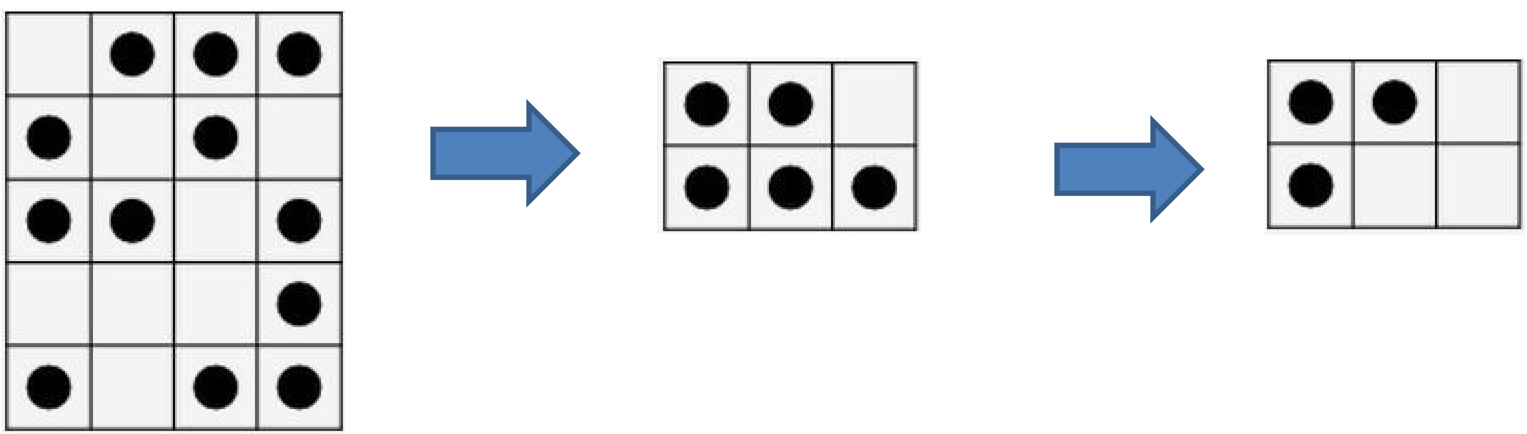
submatrix

1	1	0
1	1	1

Changing 1s to 0s

1	1	0
1	0	0

B



Extremal Function of 0-1 Matrices

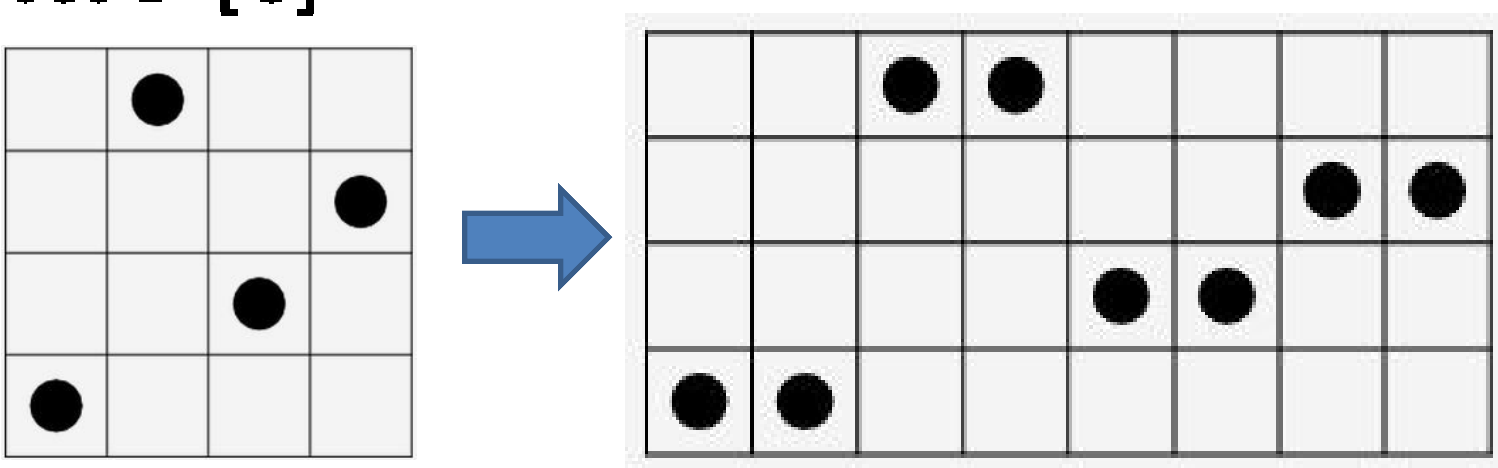
- By deleting rows and columns and changing some 1s to 0s, $\boldsymbol{A}$ **contains** $\boldsymbol{B}$.

- Otherwise we say $\boldsymbol{A}$ **avoids** $\boldsymbol{B}$

- Let $\text{ex}(n, \boldsymbol{P})$ be the maximum number of one entries in a $n \times n$ matrix avoiding $\boldsymbol{P}$

Some background

- What are all matrices $\boldsymbol{P}$ such that $\mathrm{ex}(n, \boldsymbol{P}) = O(n)$? [FH]
- $\mathrm{ex}(n, \boldsymbol{P}) = O(n)$ for permutation matrices $\boldsymbol{P}$ [MT]
- $\mathrm{ex}(n, \boldsymbol{P}) = O(n)$ for double permutation matrices $\boldsymbol{P}$ [G]

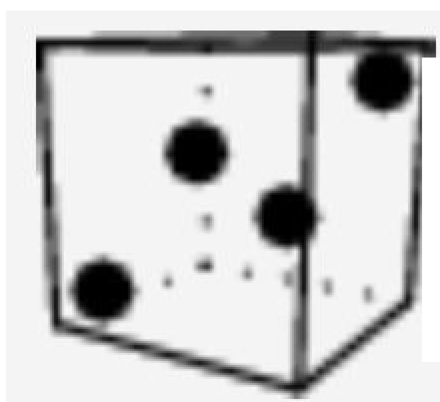


d-dimensional

- **$\boldsymbol{d}$-dimensional 0-1 matrix** $\boldsymbol{M} = (M;\, n_1, n_2, \ldots, n_d)$ where $M \subset [n_1] \times [n_2] \times [n_3] \times \cdots \times [n_d]$
- **extremal function** $f(n, \boldsymbol{P}, d)$ is the maximum number of ones in an $n \times \cdots \times n$ d-dimensional 0-1 matrix that avoids the d-dimensional matrix $\boldsymbol{P}$
- $f(n, \boldsymbol{P}, 2) = \mathrm{ex}(n, \boldsymbol{P})$

More Questions

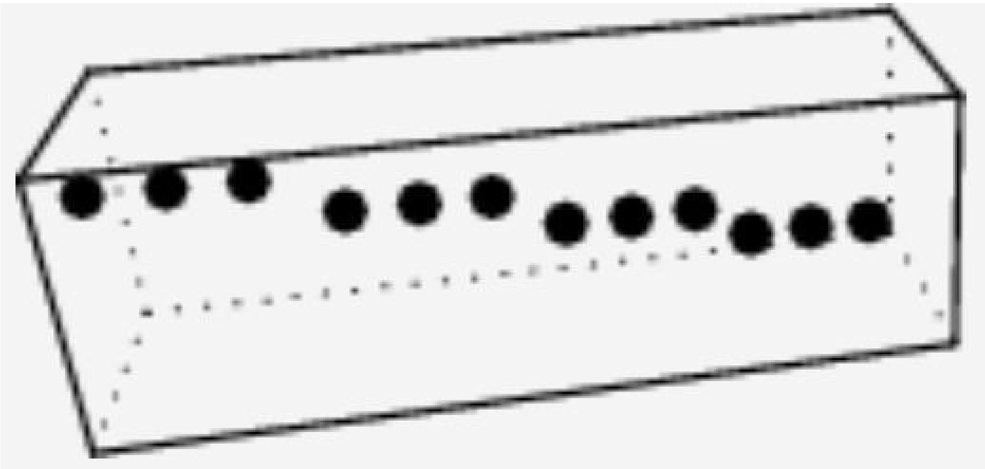
- What are all matrices $\boldsymbol{P}$ such that $f(n, \boldsymbol{P}, d) = O(n^{d-1})$?
- $f(n, \boldsymbol{P}, d) = O(n^{d-1})$ for all d-dimensional permutation matrices $\boldsymbol{P}$ [KM]



Permutation matrix with $d = 3, k = 4$

Theorem 1

- **Tuple Permutation matrix**



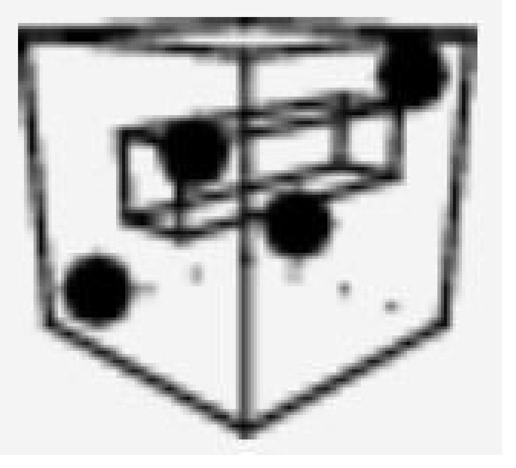
Tuple permutation matrix with $d = 3, j = 3, k = 4$

- $\boldsymbol{f(n, j, k, d)} = \max_{\boldsymbol{P}} f(n, \boldsymbol{P}, d)$, where $\boldsymbol{P}$ ranges over j-tuple permutation matrices of size k

Theorem 1: $\boldsymbol{f(n, j, k, d) = O(n^{d-1})}$.

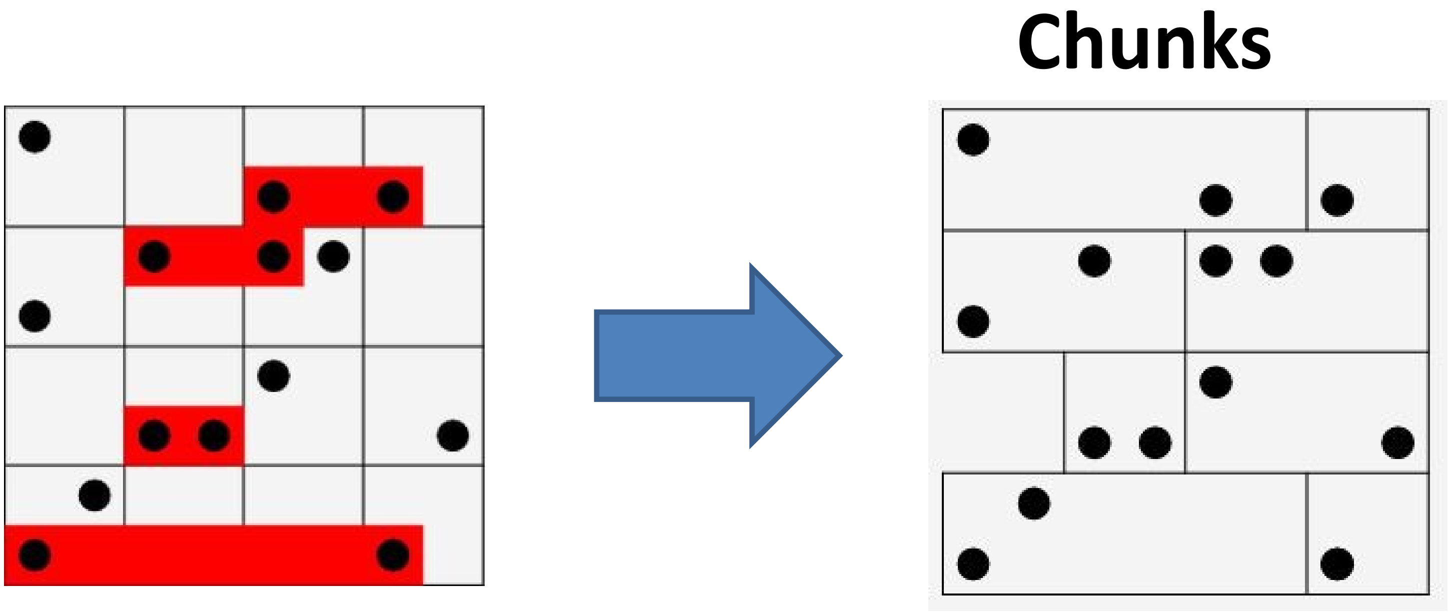
Proof of Theorem 1

- **Let $\boldsymbol{A}$ be an $sn \times \cdots \times sn$ matrix that avoids $2k \times k \times \cdots \times k$ double permutation matrix P**
- **An *$\boldsymbol{i}$*-row is a maximum set of entries $(x_1, x_2, \ldots, x_d)$ with only x_i varying**



Proof of Theorem 1

- Divide $\boldsymbol{A}$ into n^d blocks of size $s \times \cdots \times s$



$n = 4$, $s = 2$, and $d = 2$

Proof of Theorem 1

- **Wide Chunks** have at least $2k$ one entries in the same 1-row
- A wide chunk has one non-empty block

- ***j*-tall chunks** have at least k one entries with distinct coordinates in the jth dimension

Proof of Theorem 1

The maximum number of 1s in $\boldsymbol{A}$ is:

$$f(sn,2,k,d) \le s^d[n\binom{s}{2k}f(n,1,k,d-1)]$$

(max number of 1s from wide chunks in $\boldsymbol{A}$)

$$+(2k-1)s^{d-1}[(d-1)n\binom{s}{k}f\left(n,1+s^{d-2},k,d-1\right)]$$

(max number of 1s from j-tall but non wide chunks in $\boldsymbol{A}$)

$$+(2k-1)(k-1)^d[f(n,2,k,d)]$$

(max number of 1s from non-wide, non-tall chunks in $\boldsymbol{A}$)

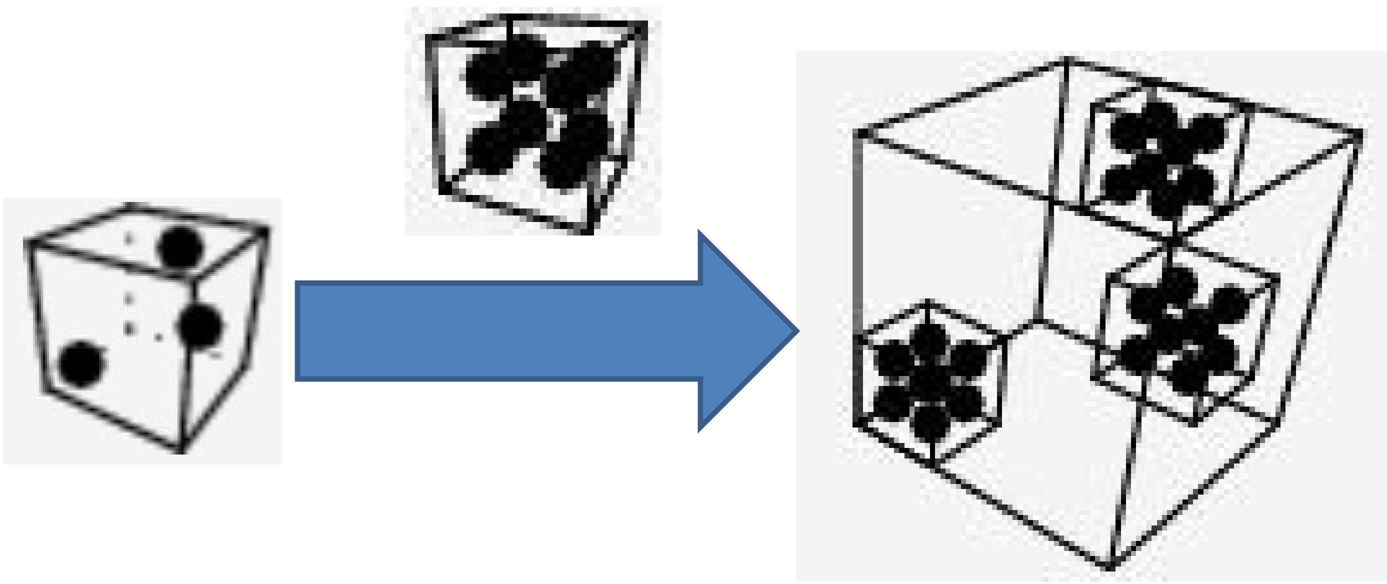
Proof of Theorem 1

- By induction on n and d, $f(n, 2, k, d) = O(n^{d-1})$
- By induction on j, $f(n, j, k, d) = O(n^{d-1})$

Block Permutation Matrices

- Block matrices $\boldsymbol{R}^{k_1,k_2,\ldots,k_d}$

- Block permutation matrices $\boldsymbol{P}^{k_1,k_2,\ldots,k_d}$



Theorem 2

Theorem 2: $f\left(n, R^{k_1,k_2,\ldots,k_d}, d\right) = O(n^{d-\frac{\max(k_1,k_2,\ldots k_d)}{k_1 k_2 \ldots k_d}})$

- **Base Case:** Kovari, Sos, and Turan proved this for $d = 2$

Lower bound on blocks

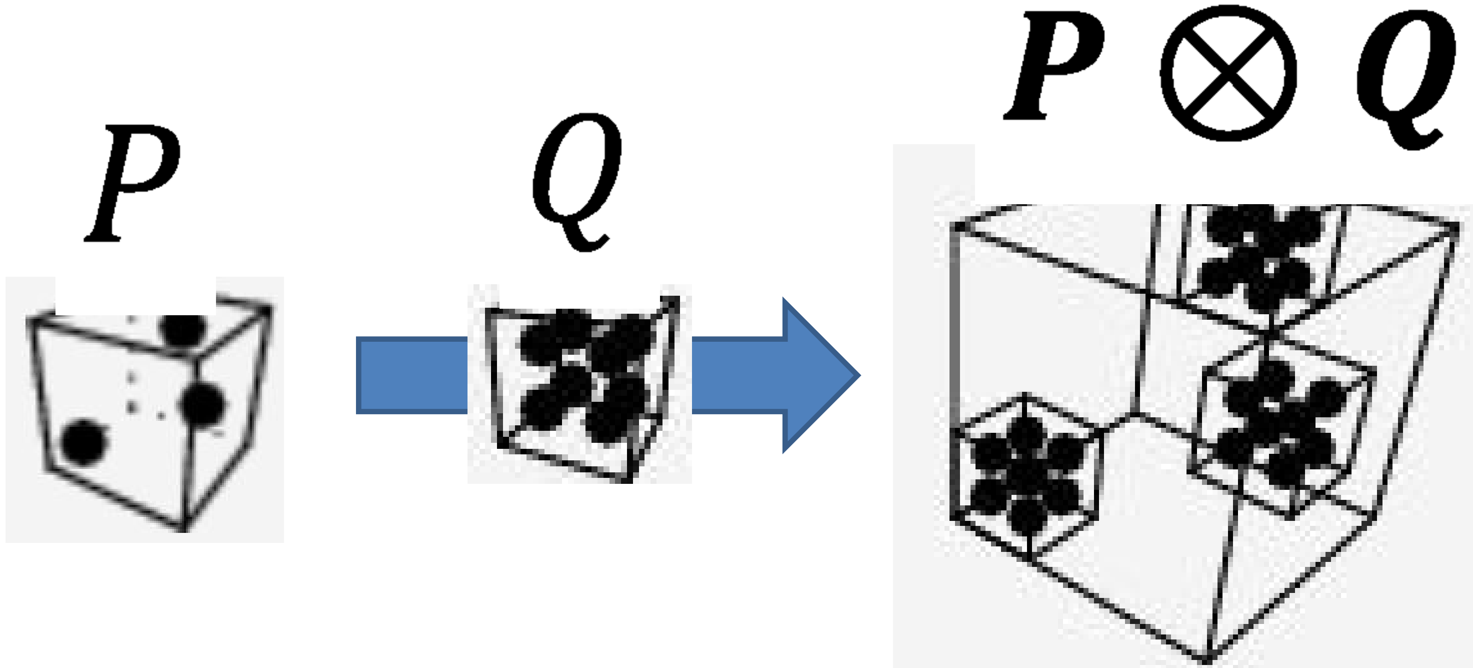
Theorem 3: $f\left(n, R^{k_1,\dots,k_d}, d\right) = \Omega\left(n^{d - \frac{k_1 + k_2 + \dots + k_d - d}{k_1 k_2 \dots k_d - 1}}\right)$.

- A function is unboundedly super n^{d-1} if for all k there exists c such that for all n, $f(cn) > kc^{d-1} f(n)$
- $n^{d-1+\epsilon}$ for $\epsilon > 0$ is unboundedly super n^{d-1}

Tensor product

- $\boldsymbol{P} \otimes \boldsymbol{Q}$ is the matrix obtained by replacing each 1 of $\boldsymbol{P}$ with a copy of $\boldsymbol{Q}$

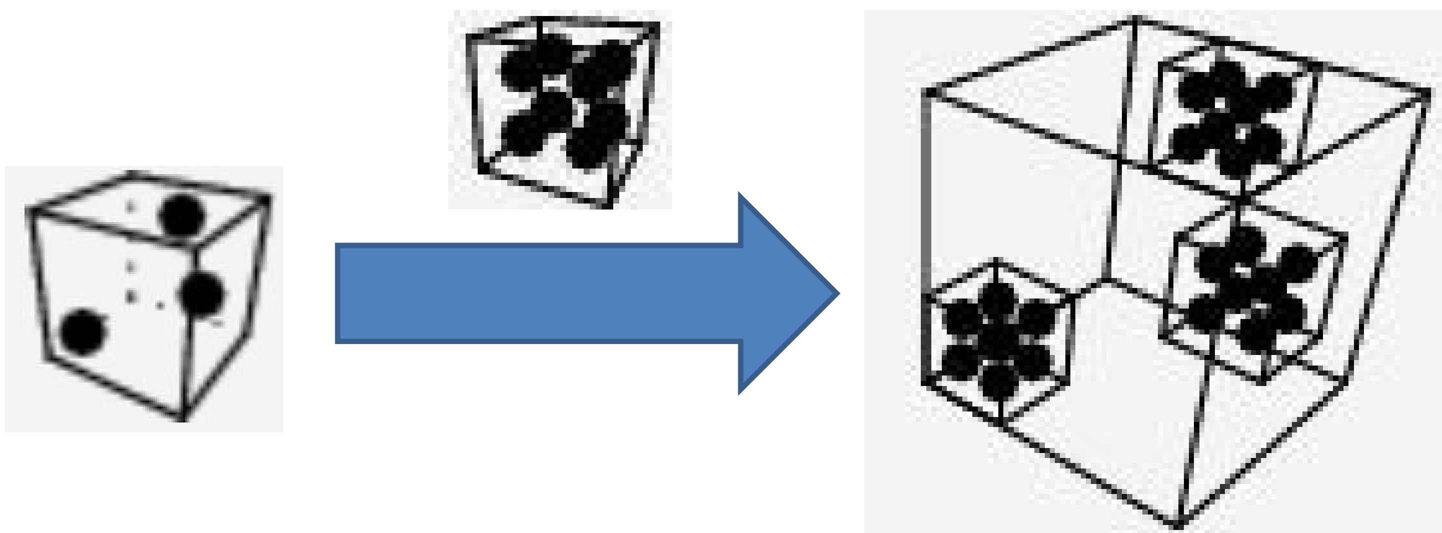
Lemma: If $\boldsymbol{P}$ is a d-dimensional permutation matrix and $\boldsymbol{Q}$ is a matrix such that $f(n, \boldsymbol{Q}, d)$ is unboundedly super n^{d-1} then $f(n, \boldsymbol{P} \otimes \boldsymbol{Q}, d) = \Theta(f(n, \boldsymbol{Q}, d))$



Block permutation matrix

- $\boldsymbol{P}^{k_1,k_2,\ldots,k_d} = \boldsymbol{P} \otimes \boldsymbol{R}^{k_1,k_2,\ldots,k_d}$

Corollary: If $\boldsymbol{P}$ is a permutation matrix, then
$f\left(n, \boldsymbol{P}^{k_1,k_2,\ldots,k_d}, d\right) = \Theta\left(f\left(n, \boldsymbol{R}^{k_1,k_2,\ldots,k_d}, d\right)\right)$



Open Problems

- What are all d-dimensional matrices $\boldsymbol{P}$ such that $f(n, \boldsymbol{P}, d) = O(n^{d-1})$?
- What are all d-dimensional matrices $\boldsymbol{P}$ such that $f(n, \boldsymbol{P}, d)$ is unboundedly super n^{d-1}?
- What are tight bounds on $f\left(n, \boldsymbol{R}^{k_1,k_2,\ldots,k_d}, d\right)$ and $f\left(n, \boldsymbol{P}^{k_1,k_2,\ldots,k_d}, d\right)$?

References

- [FH] Zoltán Füredi and Péter Hajnal, Davenport Schinzel Theory of Matrices
- [G] Jesse Geneson, Extremal functions of forbidden double permutation matrices
- [KM] Martin Klazar and Adam Marcus, Extensions of the linear bound in the Füredi-Hajnal conjecture
- [KST] Kővári, Sós, and Turán, On a problem of Zarankiewicz
- [MT] Adam Marcus and Gabor Tardos, Excluded permutation matrices and the Stanley Wilf conjecture

Acknowledgements

- My mentor Jesse Geneson
- MIT PRIMES USA
- My family