Novel Relationships Between Circular Planar Graphs and Electrical Networks

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Abstract

Circular planar graphs are used to model electrical networks, which arise in classical physics. Associated with such a network is a network response matrix, which carries information about how the network behaves in response to certain potential differences. Circular planar graphs can be organized into equivalence classes based upon these response matrices. In each equivalence class, certain fundamental elements are called critical. Additionally, it is known that equivalent graphs are related by certain local transformations. Using wiring diagrams, we first investigate the number of Y-Δ transformations required to transform one critical graph in an equivalence class into another, proving a quartic bound in the order of the graph. Next, we consider positivity phenomena, studying how testing the signs of certain circular minors can be used to determine if a given network response matrix is associated with a particular equivalence class. In particular, we prove a conjecture by Kenyon and Wilson for some cases.

1 Introduction

From classical electromagnetism, resistor networks connected to batteries have been widely studied and used in electrical engineering. In particular, such networks can be manipulated (most famously through parallel, series, and Y-Δ transformations) to understand properties of current and voltage. Circular planar graphs and electrical networks extend these well-known notions. The study of these objects was begun by de Verdière-Gitler-Vertigan [dVGV] and Curtis, Ingerman, and Morrow [CIM] who considered how the network responds (through a *response matrix*) to voltages applied to boundary vertices. A notion of equivalence between electrical networks was developed and has been understood combinatorially through *local equivalences* and *medial graphs*. Naturally, an inverse boundary problem was addressed: given information about a network (its response matrix), what data can be recovered? Aside from certain combinatorial data concerning the signs of certain minors, it was determined that for a network with an underlying graph that is *critical*, the conductances can be uniquely recovered.

Additionally, Alman, Lian, and Tran constructed a graded poset EP_n to organize equivalence classes of eletrical networks with n boundary vertices, and have analyzed various enumerative and topological properties [ALT]. In relation to response matrices, a natural question to ask is as follows: Given such a response matrix, can we efficiently determine which of electrical networks it is associated with? To this end, Kenyon and Wilson have conjectured that knowledge of the signs of certain small sets of minors can determine the equivalence class of a particular circular planar electrical network [KW]. Alman, Lian, and Tran study such sets, called *positivity tests*, for the top rank element of EP_n [ALT2] .

In Section 2 of this paper, we provide detailed definitions and background on the topic. In Section 3, we introduce a novel discussion of the size of equivalence classes, showing that the number of *Y-Δ transformations* needed to transform one critical electrical network into an equivalent one is at most quartic in the n. In Section 4, we address certain cases of Kenyon and Wilson's conjecture. We conclude in Section 5, addressing further research.

This paper addresses certain questions concerning electrical networks that are algorithmic in nature. Thus, the results have potential implications for computer science, improved circuit design, and networking efficiency. Circular planar graphs have also been widely used in a novel medical imaging technique known as electrical impedence tomography [BDGV08]. Critical circular planar graphs, due to unique properties of simplicity and recoverability, can be naturally used to model this process. Our work concerning critical graphs may be helpful for optimal modeling.

2 Definitions and Background

Definition 2.1. A **circular planar graph** $\Gamma = (V, E)$ is a planar graph embedded in a disc D with a designated set boundary vertices V_B on the boundary of D. The order n is the number of boundary vertices. Self-loops and interior vertices of degree 1 are not allowed.

Definition 2.2. A **circular planar electrical network** (Γ, γ) is a circular planar graph $\Gamma = (V, E)$ along with a conductance map $\gamma : E \to \mathbb{R}_{>0}$. See Figure 1 for an example.

This object can be interpreted physically as having resistors of conductance $\gamma(e)$ in place of edges. In particular, the network satisfies laws in classical physics: Kirchoff's law and Ohm's law. Fix some circular planar electrical network, label the boundary vertices $b_1, \ldots, b_n$, and suppose that electrical potentials $v_1, \ldots, v_n$ are assigned to these boundary vertices. This defines a natural map $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(v_1, \ldots, v_n) = (c_1, \ldots, c_n)$ where c_i is the current through boundary vertex v_i. This map f is, in fact, linear [CIM]. Using the natural bases for $\mathbb{R}^n$ we then can define a **network response matrix** for the map f associated with each circular planar electrical network (Γ, γ).

Definition 2.3. Two electrical networks are said to be **equivalent** if they have the same network response matrices.

Equivalent electrical networks are related by the following local equivalences [dVGV, Théorème 4]. These can be performed in reverse as well.

- Parallel equivalence: Two edges with common endpoints v and w and conductance a and b can be replaced by an edge of conductance $a + b$.

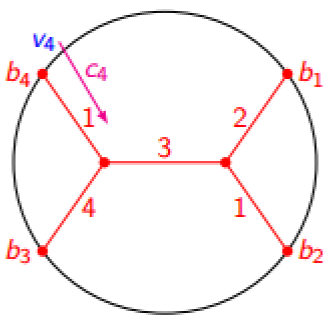

Figure 1: Electrical Network

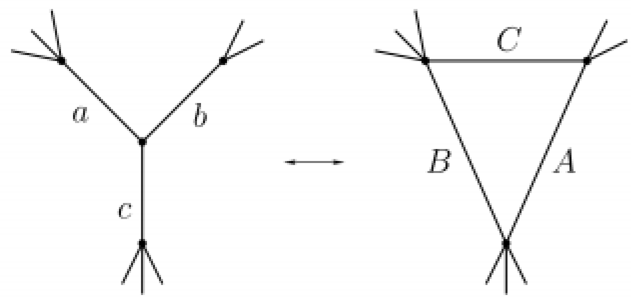

Figure 2: Y-Δ Transformation

- Series equivalence: Two consecutive edges uv and vw of conductance a and b, respectively, can be replaced by an edge uw of conductance $\frac{ab}{a+b}$.

- Y-Δ equivalence: The transformation in Figure 2 with conductances satisfying

$$A = \frac{bc}{a+b+c} \quad B = \frac{ac}{a+b+c} \quad C = \frac{ab}{a+b+c}.$$

Definition 2.4. Two circular planar graphs are said to be **equivalent** if they are related by a sequence of local equivalences (disregarding the references to conductances, of course).

For circular planar graphs in the same equivalence class, the space of possible response matrices is the same. The "simplest" elements of an equivalence class are called critical:

Definition 2.5. A circular planar graph is called **critical** if it has the least number of edges in its equivalence class.

Theorem 2.6 ([CIM, Theorem 1])**.** *Equivalent critical circular planar graphs are related by Y-Δ transformations.*

A useful dual object to circular planar graph is its *medial graph.* The construction of a medial graph $M(\Gamma)$ from a circular planar graph $\Gamma = (V, E)$ is shown in Figure 3. A medial graph can be associated with a *wiring diagram* (as in Figure 4). More detailed description and theory can be found in [CIM] and [ALT]. For our purposes, it will suffice to know that *critical* circular planar graphs are in bijection with full, lensless wiring diagrams:

Definition 2.7. Fix a disc D with n boundary vertices. For each boundary vertex, place two medial boundary vertices, one on each side. A **wiring diagram** is a collection of n smooth curves embedded in a disc D, such that each wire connects two medial boundary vertices. Each medial boundary vertex is associated with one wire, and triple crossings and self-loops are not allowed. The **order** of W is defined to be n. We call W **lensless** if any two wires intersect at most once, and we call W **full** if no two boundary vertices can be connected by a smooth curve inside D without intersecting a wire.

Theorem 2.8 ([CIM, Lemma 13.1], [ALT, Theorem 2.4.6])**.** *Critical circular planar graphs of order n are in bijection with full, lensless wiring diagrams of order n.*

Proposition 2.9 ([CIM])**.** *If two critical circular planar graphs are related by a Y-Δ transformation, then their corresponding wiring diagrams are related by a transformation known as a* ***motion****, shown in Figure* 5*.*

(A motion can be thought of as lifting a wire over the intersection of two other wires).

Therefore, equivalence classes of critical graphs under Y-Δ transformations are isomorphic to equivalence classes of wiring diagrams under motions. The following result characterizes such equivalence classes.

Theorem 2.10 ([CIM, Theorem 7.2])**.** *If W_1 and W_2 are full, lensless wiring diagrams whose wires pair up the medial boundary vertices in the same manner, then W_1 and W_2 are related by a sequence of motions.*

We will make use of the following construction from [ALT, Lemma 3.1.1]:

Definition 2.11. Circular planar graph equivalence classes can be organized into a poset EP_n with ordering relation as follows: if G can be transformed into H by contracting and/or deleting some sequence of edges, then $[G] \geq [H]$.

Theorem 2.12 ([ALT, Theorem 3.2.4])**.** *EP_n is graded by the number of edges in the critical graph of the equivalence classes. In the top rank element, graphs have $\binom{n}{2}$ edges.*

We can interpret this poset as one of equivalence classes of full, lensless wiring diagrams.

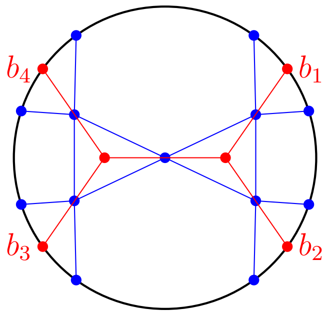

Figure 3

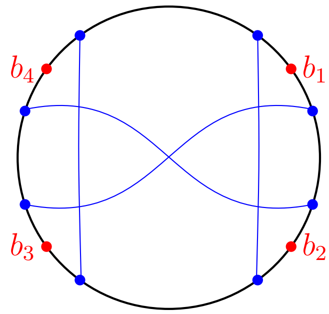

Figure 4

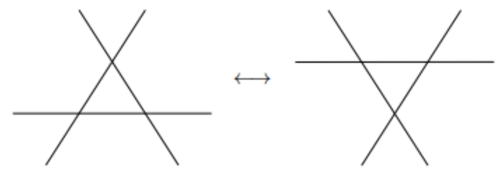

Figure 5

Shown in Figure 3 is a circular planar graph of order 4 (red) along with its medial graph (red). The corresponding wiring diagram is given in Figure 4. Figure 5 illustrates a motion.

Contractions and deletions of edges in circular planar graphs correspond to breaking the crossings of two wires in the wiring diagram. In particular, if a, b, c, d are medial boundary vertices in clockwise order around the circle and ac and bd are two wires, then we can break the crossing by replacing these wires with either wires ab and cd or ad and bc.

An interesting combinatorial characterization of circular planar graphs involves circular pairs and circular minors:

Definition 2.13. A **circular pair** $(P; Q)$ is an ordered pair of two sequences of distinct vertices $P = (p_1, \ldots, p_m)$ and $Q = (q_1, \ldots, q_m)$ such that $p_1, \ldots, p_m, q_m, \ldots, q_1$ appear in clockwise circular order.

For $n = 6$, an example of a circular pair is $(1, 2; 6, 4)$.

Definition 2.14. Let Γ be a circular planar graph. A circular pair (P, Q) is said to be **connected** if there exist pairwise vertex-disjoint paths from p_i to q_i. The path from p_i to q_i must not use any other boundary vertices.

Definition 2.15. Let (Γ, γ) be an electrical network, let $(P; Q)$ be a circular pair, and let N be the response matrix. The **circular minor** associated with $(P; Q)$ is the determinant of the submatrix of N with ordered row set P and ordered column set Q.

Theorem 2.16 ([CIM, Theorem 4, Theorem 4.2])**.** *All circular minors of the response matrix are non-negative. Positive circular minors correspond to connected circular pairs in* Γ.

3 Critical Graph Equivalences Classes

In this section, we will prove the following:

Theorem 3.1. *Suppose Γ_1 and Γ_2 are equivalent critical circular planar graphs of order n. The number of Y-Δ equivalences required to transform Γ_1 into Γ_2 is at most quartic in n.*

This can be rephrased as: Given two full, lensless wiring diagrams W_1 and W_2 of order n whose wires pair up the same vertices, the number motions are needed to transform one into the other is at most quartic in n.

We first consider the equivalence class M_n of wiring diagrams corresponding to the top rank element of EP_n, with $\binom{n}{2}$ intersections. Number the medial boundary vertices 1 to $2n$ in counterclockwise order. Our wiring diagram must match the medial boundary vertex i to $i+n$ modulo $2n$. We now relate elements of M_n to reduced decompositions of the permutation $(n, \ldots, 1) \in S_n$. Write t_x for the transposition $(x, x+1)$.

Definition 3.2. A **reduced decomposition** of a permutation $\sigma \in S_n$ is a sequence of transpositions of the form t_x such that $\sigma = s_1 \cdots s_k$ under composition and k is minimal.

Fix a reduced decomposition $s_1 \cdots s_k$ of $(n, \ldots, 1)$. It is well-known that $k = \binom{n}{2}$.

Definition 3.3. If $s_i = t_x$ and $s_{i+1} = t_y$ with $|x - y| \geq 2$, they can be swapped to produce another valid reduced decomposition $s_1 \cdots s_{i-1} s_{i+1} s_i s_{i+2} \cdots s_n$. This is called a **2-move**.

Definition 3.4. If $s_{i-1} = s_{i+1} = t_x$ and $s_i = t_y$ where $|y - x| = 1$, replacing the sequence $s_{i-1} s_i s_{i+1} = t_x t_y t_x$ with $t_y t_x t_y$ does not alter the resulting permutation. This is a **3-move**.

Call two reduced decompositions of $(n, n-1, \ldots, 1) \in S_n$ 2-*equivalent* if a sequence of 2-moves can transform one into the other. Let R_n be the set of these equivalence classes.

Lemma 3.5. *R_n and M_n are in natural bijection, and 3-moves in R_n correspond to motions in M_n.*

Proof. Fix a wiring diagram in M_n (as in Figure 6). Discard the circle and pull the vertices labeled 1 to n to one side and the vertices labeled $n+1$ to $2n$ to the other side. We organize the wires as shown in Figure 7, with intersections occurring in one of the $n-1$ horizontal

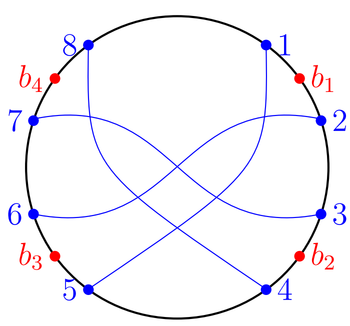

Figure 6

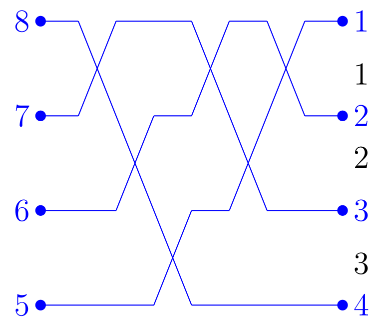

Figure 7

Figure 6 shows a wiring diagram in M_4 and Figure 7 shows the result of deforming the wiring diagram.

rows in a well-defined order. Label i the row between vertex i and $i+1$. Define a reduced decomposition $s_1 \cdots s_k$ as follows: Recording from left to right, if the ith intersection occurs in row x, write $s_i = t_x$. Indeed, any 2-move on the resulting decomposition corresponds to a homotopy on the wiring diagram. Any 3-move corresponds to a motion. $\square$

For a reduced decomposition r of $(n, \ldots, 1)$, let $[r] \in R_n$ denote its equivalence class under 2-moves. Define $F([r]) = (a_1, a_2, \ldots, a_{n-1})$, where a_x denotes the number of transpositions of the form t_x in r.

Lemma 3.6. *Let $[m] \in R_n$ be such that a sequence of the form $t_{x+1}t_xt_{x+1}$ does not appear in any element of the equivalence class. Then $F([m]) = (n-1, \ldots, 1)$.*

Proof. We proceed by induction on n. For $n = 2$, this is trivial. Suppose it holds for $n-1$.

Let $m = s_1 \cdots s_{\binom{n}{2}}$ We construct a sequence of indices $i_1, \ldots, i_{n-1}$ as follows. Let i_{n-1} be the largest index for which $s_{i_{n-1}} = t_{n-1}$. Let i_k be the largest index less than i_{k+1} such that $s_{i_k} = t_k$. This is a well-defined sequence because when $s_1 \cdots s_k$ is applied to the permutation $(1, \ldots, n)$, the transpositions that change the position of the number n will be $s_{i_1}, \ldots, s_{i_{n-1}}$.

We now inductively apply two moves so that $i_k, \ldots, i_{n-1}$ are consecutive numbers. Suppose for some k this is true. Let $s_{i_k - 1} = t_x$. If $x = k-1$, we have $i_{k-1}, \ldots, i_{n-1}$ are consecutive numbers. We cannot have $x = k$ because this would imply m is not a reduced decomposition. Suppose that $x > k$. Then, a sequence of 2-moves can be applied so that $t_xt_{x-1}t_x$ shows up in the reduced decomposition, a contradiction. If $x < k-1$, we can apply a sequence of 2-moves so that the sequence $i_k, \ldots, i_{n-1} \mapsto i_k - 1, \ldots, i_{n-1} - 1$ and i_{k-1} remains fixed. Therefore, by induction, we can ensure $i_1, \ldots, i_{n-1}$ are consecutive. By a similar argument as above, we must also have that $i_k = k$ for all k.

Now, $r = s_n \cdots s_{\binom{n}{2}}$ is a reduced decomposition of the permutation $(n-1, \ldots, 1)$ such that a consecutive sequence of the form $t_{x+1}t_x t_{x+1}$ never appears in any element of $[r]$. Using the induction hypothesis, we can see that $F([m]) = (n-1, \ldots, 1)$, as desired. $\square$

Corollary 3.7. *The lexicographically highest output of F is $(n-1, n-2, \ldots, 1)$. A reduced decomposition r for which $F([r])$ is not lexicographically maximal can be transformed via 3-moves (and 2-moves) to a reduced decomposition r' where $F([r'])$ is lexicographically higher.*

Lemma 3.8. *If $F([r]) = F([r']) = (n-1, n-2, \ldots, 1)$, then $[r] = [r']$.*

Proof. This follows from using 2-moves in a similar way as in Lemma 3.6. $\square$

Define $G([r]) = (n-1, \ldots, 1) \cdot F([r])$ (dot product between vectors in $\mathbb{R}^n$).

Lemma 3.9. *Let r and r' be two reduced decompositions. Then, there exists a sequence of 2-moves and 3-moves transforming r to r' with the number of 3-moves at most cubic in n.*

Proof. Note that every time a 3-move of the form $t_{x+1}t_x t_{x+1} \mapsto t_x t_{x+1} t_x$ is applied, the value of $G([r])$ increases by 1. Let r_0 be a reduced decomposition such that $F([r]) = (n-1, \ldots, 1)$. By Corollary 3.7 and Lemma 3.8, we can transform r into r_0 with 2-moves and 3-moves of the form $t_{x+1}t_x t_{x+1} \mapsto t_x t_{x+1} t_x$. Similarly, we can transform r_0 to r' with a sequence of 2-moves and 3-moves of the form $t_x t_{x+1} t_x \mapsto t_{x+1} t_x t_{x+1}$. Thus, we have $r \mapsto r_0 \mapsto r'$ using at most $2 \times (n-1, n-2, \ldots, 1) \cdot (n-2, n-4, \ldots, 2-n)$ 3-moves. This is cubic in n. $\square$

Corollary 3.10. *Let W_1 and W_2 be two wiring diagrams in M_n. Then, the number of motions required to transform W_1 to W_2 is at most cubic in n.*

We are now ready to prove Theorem 3.1 in full. We will generalize the above result to prove that two full, lensless wiring diagrams in the same equivalence class are related by at most a quartic (in n) number of motions.

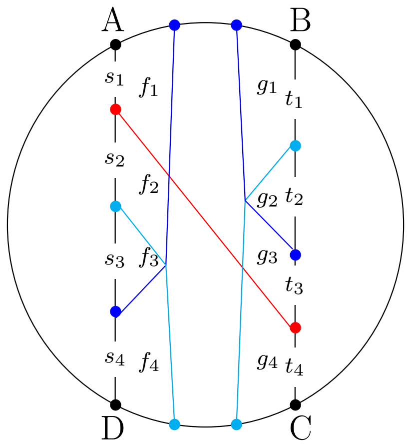

Figure 8

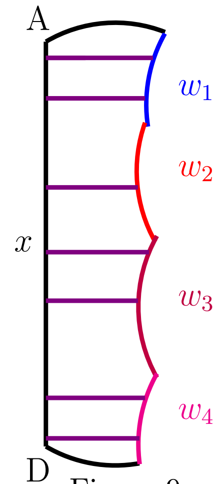

Figure 9

Figure 8 displays a portion of a wiring diagram within the region R. The wires x and y are composed of segments of the form s_i and t_i, respectively. The corresponding faces of the form f_i and g_i have also been identified. Figure 9 illustrates region F of a hypothetical wiring diagram. Note the labeling of the wire segments with w_i.

Let k be the rank of $[W_1]$ in EP_n. Call two wires *parallel* if they do not intersect. In order prove Theorem 3.1, we will show the following:

Lemma 3.11. *Some two parallel wires in W_1 can be made to border the same face in at most k motions.*

Let x and y be two parallel wires such that there is no wire z between them such that z is mutually parallel to x and y. Let A and D be the boundary vertices of x, and B and C the boundary vertices for y such that A, B, C, D are in clockwise order. Let s_1, s_2,$\ldots$,s_p be the segments of wires whose union is x such that for $i < j$, s_i is closer than s_j to A along wire x. Similarly, define, $t_1, t_2, \ldots, t_q$ to be segments of wires with union y and $i < j$, t_i is closer than t_j to B along wire y. Let R denote the region between x and y. For $1 \leq i \leq p$, let f_i be the face contained in R that has s_i as an edge; for $1 \leq j \leq q$, let g_i be the face contained in R that has t_i as an edge. (See Figure 8.) Suppose that a face f_i or g_j is a triangle. Then, through a sequence of at most k motions of x and y, we can ensure that each f_i and g_j is not a triangle. This is because two motions are never made over the same point.

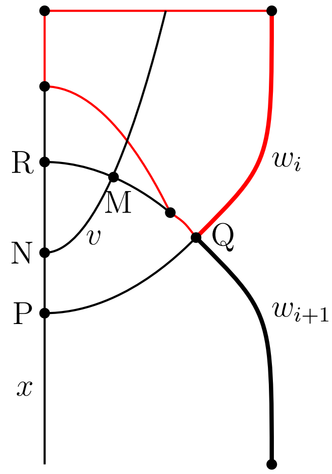

Figure 10

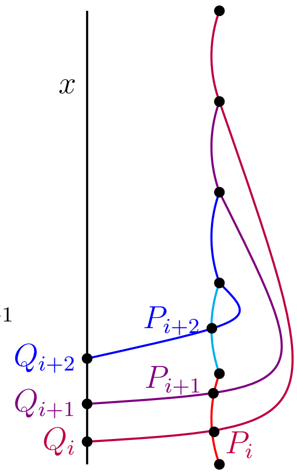

Figure 11

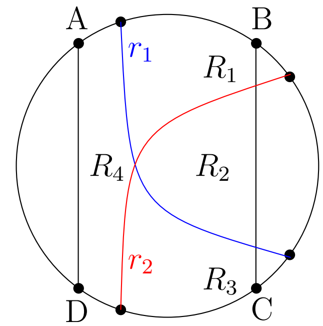

Figure 12

Figure 10 depicts the case where PQ and w_i are consecutive. Note that the wire v must partition the face outlined in red. Figure 11 illustrates the case where w_i and P_iQ_i are not consecutive. Note how wires containing w_{i+2} and w_{i+3} (blue and cyan) intersect twice. In Figure 12, the wires r_1 and r_2 partition region R into the four regions R_1, R_2, R_3, and R_4.

At this point, we claim that x and y are edges to a common face. Suppose this is not true. Let F be the union of the faces f_i. Then, the *boundary* of F consists of wire x from D to A; an arc of the boundary circle between A and B; a sequence of wire segments $w_1, w_2, \ldots, w_k$ such that two consecutive wire segments are from different wires; and an arc of the boundary circle between C and D. (See Figure 9.)

Lemma 3.12. *A wire w containing w_i cannot intersect x.*

Proof. Let P be such a point of intersection and let Q be a point on w such that Q lies on the boundary of F, the wire segment PQ is contained in region F, and P and Q are the only points on the segment PQ that are on the boundary of F. We have two cases to consider, depending on whether w_i and PQ are consecutive segments.

First, assume they are consecutive. Then Q is an endpoint of w_i and is either $w_i \cap w_{i-1}$ or $w_i \cap w_{i+1}$. Assume it is the latter. Then we can see that the wire containing w_{i+1} will intersect x at a point R, such that R is closer to A than P. Clearly PQR is a triangle and must be partitioned by at least one wire. Let v be such a wire and assume that v intersects QR at a point M. Let N be the other point of intersection with triangle PQR. Let L and O

be the boundary vertices of v such that the vertices L, M, N, and O appear in alphabetical order on v. (For example, M is closer to L than N.) We can assume that segment LM does not intersect x. Either N is on PR and therefore on x or N is on PQ. In the latter case, either NO or LM does not intersect x, and we can assume without loss of generality that LM does not. Now, consider the face in F that has Q as a vertex and a subset of w_i as an edge. Clearly, LM must partition this face, a contradiction. Thus, the segments PQ and w_i cannot be consecutive on w. (See Figure 10.)

Let $X_i = w_i \cap w_{i+1}$ for $1 \leq i \leq k-1$. So w_i can be written as $X_{i-1}X_i$. We will now consider the case where PQ and w_i are not consecutive. For a wire w containing w_i, we will write the points P and Q as P_i and Q_i, respectively. There exists a unique index $j \neq i$ such that $Q_i \in w_j$. Assume without loss of generality that P_i is closer to X_i than X_{i-1} along w. We can see that $j > i$; otherwise, w forms a loop or partitions the face in F to which X_i is a vertex. Then, it can be seen that the wire containing w_{i+1} must intersect x at point P_{i+1} such that P_{i+1} is closer to A than P_i along x; otherwise, the wires of w_i and w_{i+1} intersect twice. We also then have $Q_{i+1} \in w_l$ such that $l \leq j$. But, inductively, this is clearly impossible. (See Figure 11.) Therefore, any wire containing w_i cannot intersect x. $\square$

Proof of Lemma 3.11. We call a wire a **Type 1 wire** if it has a boundary vertex between A and B and a boundary vertex between B and C. A wire is a **Type 2 wire** if it has a boundary vertex between B and C and one between C and D. Due to Lemma 3.12, we have that each segment w_i belongs to a Type 1 or 2 wire. We can now see that there exists an index m such that wires containing w_i for $1 \leq i \leq m$ are Type 1 and otherwise are Type 2. Indeed, if w_i belongs to a Type 1 wire, then w_{i-1} must have a boundary vertex between A and B and thus must be Type 1.

Now, define a **Type 3 wire** to have a boundary vertex between A and B and one between D and A. Define a type **Type 4 wire** to have boundary vertex between C and D and one between D and A. Let G be the union of the faces g_i. For $1 \leq i \leq 4$, Let r_i be a Type i wire with the following properties:

1. Segments of r_1 and r_2 and $r_1 \cap r_2$ are on the boundary of F.

2. Segments of r_3 and r_4 and $r_3 \cap r_4$ are on the boundary of G.

We now show that r_1, r_2, r_3, r_4 cannot be arranged without violating a necessary condition. Let R_1, R_2, R_3, and R_4 be the regions into which R is partitioned by r_1 and r_2, as shown in Figure 12. Suppose $r_3 \cap r_4 \in R_4$. Consider the face in F that has $r_1 \cap r_2$ on its boundary. This face is clearly partitioned by r_3 and r_4, a contradiction. So, we assume that $r_3 \cap r_4 \in R_1 \cup R_2$. Then, r_4 and r_1 intersect twice, a contradiction. We have proven Lemma 3.11. □

Proof of Theorem 3.1. Now, let $[W]$ denote the equivalence class obtained from $[W_1]$ by matching a to c and matching b to d. Let C be the maximum number of motions required to transform one wiring diagram into another in $[W]$. We can transform W_1 to $W_1^{'}$ in at most k motions, and similarly W_2 to $W_2^{'}$, where $W_1^{'}$ and $W_2^{'}$ have parallel wires x and y as edges to a common face (by Lemma 3.11). Let $W_1^{''}$ and $W_2^{''}$ be the wiring diagrams obtained from $W_1^{'}$ and $W_2^{'}$, respectively, by crossing the segments of x and y that are common to a face. Consider a sequence of motions to transform $W_1^{''}$ to $W_2^{''}$. We can apply this sequence to $W_1^{'}$ to obtain $W_2^{'}$ by treating any motion over the newly created intersection of the wires A to C and B to D as homotopy. Therefore, we have a sequence of motions of length at most $C + 2k$ that can transform W_1 to W_2. Finally, using Corollary 3.10 and the fact that EP_n has a quadratic number of ranks, we are done. □

4 Positivity Tests

We investigate *positivity tests* for response matrices of particular equivalence classes. Let M be a symmetric n x n matrix. A question of interest is as follows: Can we determine if M is a network response matrix for some electrical network in a given equivalence class of EP_n?

Definition 4.1. An r**-positivity test** for an equivalence class $[G]$ of EP_n of corank r is defined to be an ordered pair of sets of circular minors (S_1, S_2) such that: if the elements of S_1 are positive and the elements of S_2 are 0, then M is a valid response matrix for an electrical network with underlying graph in $[G]$.

In general, the following has been conjectured:

Conjecture 4.2 ([KW])**.** *For each equivalence class of corank r, there exists an r-positivity test such that $|S_1| = \binom{n}{2} - r$ and $|S_2| = r$.*

[ALT2] shows this conjecture for $r = 0$. We will prove this conjecture for odd n and $r = 1, 2$. For simplicity, we will interchange the use and notation of circular pairs and circular minors where the context is clear. As a matter of notation, we will write the sequence of vertices $(x, x+1, \ldots, x+y-1, x+y+1, \ldots, x+z)$ as $(x, \ldots, x+z)^{x+y}$. Let $n = 2k+1$. We first describe which circular pairs are connected in equivalence classes in EP_n of corank 1 and 2.

Definition 4.3. A circular pair $(P; Q)$ is said to be **maximal** if exactly one vertex $v \in \{1, \ldots, n\}$ does not appear in P or Q.

We can identify maximal circular pairs by the vertex that has not been included. For brevity, we will write (v) for the maximal circular pair in which v does not appear.

Definition 4.4. A **rotation** of a circular pair $(P; Q)$ by c means we add c to each vertex in the circular pair modulo n.

The following provides a description of equivalences class of corank 1 and 2. Circular planar graphs in equivalence classes of corank 1 have exactly 1 maximal circular pair not connected, and all other circular pairs are connected. Circular planar graphs in equivalence classes of corank 2 have all circular pairs connected except one of the following sets of circular pairs are not connected, up to a rotation:

1. $(k+1)$ and (v), for $v \neq 1, 2k+1$
2. $(1, \ldots, k); (k+1, \ldots, 2k+1)^p$ where $k+1 \leq p \leq 2k+1$
3. $(1, \ldots, k+1)^p; (k+2, \ldots, 2k+1)$ where $1 \leq p \leq k+1$
4. $(k+1)$ and (1) and $(2, \ldots, k); (k+2, \ldots, 2k+1)^p$ where $k+2 \leq p \leq 2k+1$.

This is seen by considering contractions and deletions of the well-connected network in [CIM, Page 132]. We will now develop positivity tests for these equivalence classes.

Definition 4.5. Fix any matrix M. Suppose the row set and column set are indexed as I and J. Then, we will write $\Delta^{i_1 i_2 \cdots i_s, j_1 j_2 \dots j_t}$ for the determinant of the submatrix M' obtained by deleting rows $i_1, \dots, i_s \in I$ and columns $j_1, \dots, j_t \in J$, assuming M' is square.

Proposition 4.6. *We will use the following well-known **Grassmann-Plücker relations**:*

(a) Fix an $n \times n$ matrix M. Let a, b be elements of its row set, with row a above row b, and let c, d be elements of the column set, with column c to the left of column d. Then,

$$\Delta^{a,c}\Delta^{b,d} = \Delta^{a,d}\Delta^{b,c} + \Delta^{ab,cd}\Delta^{\emptyset,\emptyset}. \tag{*}$$

(b) Fix an $(n+1) \times n$ matrix M. Let a, b, c be rows (in this order top to bottom). Let d be a column. Then,

$$\Delta^{b,\emptyset}\Delta^{ac,d} = \Delta^{a,\emptyset}\Delta^{bc,d} + \Delta^{c,\emptyset}\Delta^{ab,d}. \tag{**}$$

Definition 4.7. A circular pair $(P; Q)$ is said to be **solid** if both $p_1, \dots, p_m$ and $q_1, \dots, q_m$ show up consecutively around a circle. For two vertices v and w on the circle, we write $d(v, w)$ for the number of boundary vertices between them moving clockwise from v to w. A solid circular pair $(P; Q)$ is called **diametric** if $|d(q_1, p_1) - d(p_m, q_m)| \leq 1$.

Let $\mathcal{D}$ denote the set of diametric pairs. It is shown in [ALT2, Corollary 4.1.9] that $(\mathcal{D}, \emptyset)$ is a 0-positivity test. We first give a 2-positivity test for case 4 above. Define the following:

$A = (k+1)$	$B = (1)$
$C = (2, \dots, k); (k+2, \dots, 2k)$	$D = (2, \dots, k); (k+3, \dots, 2k+1)$
$E = (1, \dots, k+1)^2; (k+2, \dots, 2k+1)$	$F = (1, \dots, k)^2; (k+2, \dots, 2k)$
$G = (1, \dots, k)^2; (k+3, \dots, 2k+1)$	$X = (1, \dots, k-1); (k+3, \dots, 2k+1)$
$Y = (3, \dots, k+1); (k+2, \dots, 2k)$	

Lemma 4.8. *If $S_1 = \mathcal{D} \cup \{X, E\} \setminus \{A, B, C, D\}$, $S_2 = \{C, D\}$, then (S_1, S_2) is a 2-positivity test.*

Proof. Suppose the elements of S_1 are positive and those of S_2 are 0. We first show that $A = B = 0$. Consider the following uses of the first Plucker relation.

	Δ	a	b	c	d
(*)(1)	$(1,\dots,k);(2k+1,\dots,k+2)$	1	k	$2k+1$	$k+2$
(*)(2)	$(2,\dots,k+1);(2k+1,\dots,k+2)$	2	$k+1$	$2k+1$	$k+2$

In (1), using $\Delta^{a,d} = \Delta^{a,c} = 0$ and $\Delta^{ab,cd} > 0$, we have $A = 0$. In (2), using $\Delta^{b,d} = \Delta^{b,c} = 0$ and $\Delta^{ab,cd} > 0$, $B = 0$.

We now show $Y > 0$. We use the following three instances of the Plücker relations.

	Δ	a	b	c	d
(**)(3)	$(1,\dots,k);(2k+1,\dots,k+3)$	1	2	k	$2k+1$
(*)(4)	$(1,\dots,k+1)^2;(2k+1,\dots,k+2)$	1	$k+1$	$2k+1$	$k+2$
(**)(5)	$(1,\dots,k);(2k,\dots,k+2)$	1	2	k	$k+2$

From (3), we can deduce $G > 0$. From (5), we then have $F > 0$. Together with (4), $Y > 0$ as desired.

We will now show that the remainder of the circular minors associated with solid circular pairs are positive. In particular, we will use strong induction on the value $e(P;Q) = |d(q_1,p_1) - d(p_m,q_m)|$. Fix some such $(P;Q) \notin \mathcal{D} \cup \{A,B,C,D,X,Y\}$. Note that $e(P;Q) \geq 3$. Suppose for all solid circular pairs $(P';Q')$ with $e(P';Q') < e(P;Q)$, we have deduced the desired sign of the associated circular minor, either 0 or positive. We assume without loss of generality that $d(q_1,p_1) < d(p_m,q_m)$. Then consider the following relation:

	Δ	a	b	c	d
(*)(6)	$(p_1,\dots,p_m,p_m+1);(q_1,\dots,q_m,q_m+1)$	p_1	p_m+1	q_1	q_m+1

We wish to show $\Delta^{b,d} > 0$. Note that the rest of the terms in the relation correspond to solid circular pairs with lower values of e. Since $(P;Q) \notin \mathcal{D} \cup \{A,B,C,D,X,Y\}$, we can deduce that $\Delta^{a,c} > 0$. Now, suppose in (6), both products on the right hand side of the relation are 0. This implies $\Delta^{b,d} = 0$. This would mean that in any circular planar graph with circular pairs in S_1 connected $(P;Q)$ would not be connected. However, the existence of the equivalence class described in case 4 above gives us a contradiction. Since each term on the right hand side of (6) is nonnegataive by the induction hypothesis, the right hand side must be positive. Therefore, we have that $\Delta^{b,d} > 0$. By induction, we have deduced signs of all solid circular pairs.

We show minors of the form $(2, \ldots, k); (2k+1, \ldots, k+2)^p$ for $k+3 \leq p \leq 2k$ are 0.

	Δ	a	b	c	d
(**)(7)	$(2, \ldots, k); (2k+1, \ldots, k+2)^p$	1	2	$k+1$	p

Since $A = B = 0$ and $E > 0$, we have $\Delta^{ac,d} = 0$.

Definition 4.9. Let $(P; Q)$ be a circular pair. Define $\phi(P; Q) = d(p_1, p_m) + d(q_m, q_1)$ if $(P; Q)$ is not solid. Otherwise, let $\phi(P; Q) = 0$.

Definition 4.10. Call a circular pair $(P; Q)$ **one-holed** if one of P and Q consists of consecutive vertices and the other can be made into a consecutive sequence of vertices with the addition of a single vertex.

Let S be the set of one-holed circular pairs with $\phi(P; Q) \leq 2k - 1$, not including circular pairs of the form $(2, \ldots, k); (2k+1, \ldots, k+1)^p$. We show using induction on ϕ that minors associated with elements of S are positive. Fix $(P; Q) \in S$ and assume that the result is true for all circular pairs with smaller associated ϕ. Assume that P is one-holed and let us label the vertex that is not included with p, such that p is between p_c and p_{c+1} on the circle.

	Δ	a	b	c	d
(**)(8)	$(p_1, \ldots, p_c, p, p_{c+1}, \ldots, p_m); (q_1, \ldots, q_m)$	p_1	p	p_m	q_1

In (8), $\Delta^{b,\emptyset} = (P; Q)$. It can be easily checked that $\Delta^{ac,d}, \Delta^{bc,d}$, and $\Delta^{ab,d}$ are either in S with a lower value of ϕ or are among solid circular pairs with arc size less than $k-1$. Each is positive. Also, we observe that $\Delta^{a,\emptyset}$ and $\Delta^{c,\emptyset}$ are both solid and at least one must be positive. Thus, the induction is complete.

We will now show that circular minors associated with the pairs $(k+1, \ldots, 2k+1)^p; (k, \ldots, 1)$ where $2k \geq p \geq k+2$ and $(k+2, \ldots, 2k+1, 1)^p; (k+1, \ldots, 2)$ where $2k+1 \geq p \geq k+3$ are positive. We use the following:

	Δ	a	b	c	d
(**)(9)	$(k+1,\dots,2k+1);(k,\dots,1)$	$k+1$	p	$2k+1$	k
(**)(10)	$(k+2,\dots,2k+1,1);(k+1,\dots,2)$	$k+2$	p	1	2

In (9), $\Delta^{ac,d}$ and $\Delta^{c,\emptyset}$ are positive minors associated with diametric pairs. Also, $\Delta^{ab,d} \in S$ and $\Delta^{a,\emptyset} = 0$. Thus, $\Delta^{b,\emptyset} > 0$. Similarly, in (10), we conclude $\Delta^{b,\emptyset} > 0$. Now, we show positivity of minors of one-holed circular pairs of the form $(P;Q) = (1,\dots,k+1)^p,(2k+1,\dots,k+2)$. For $p=2$, $(P,Q) = E \in S_1$. We proceed by induction on p from 2 to k to show each such $(P;Q)$ is positive. Assume for some $2 \le p \le k-1$ we have that the minor of $(1,\dots,k+1)^p;(2k+1,\dots,k+2)$ is positive.

	Δ	a	b	c	d
(**)(11)	$(1,\dots,k+1);(2k+1,\dots,k+2)$	1	p	$p+1$	$2k+1$

In (11), we have by the induction hypothesis $\Delta^{b,\emptyset} > 0$. Additionally, $\Delta^{a,\emptyset} = B = 0$ and $\Delta^{ab,d}$, $\Delta^{ac,d} \in S \cup \{Y\}$. Thus, $\Delta^{c,\emptyset} > 0$, as desired in our induction.

Finally, we can finish the proof by showing positivity of the remainder of minors of non-solid circular pairs. We proceed by induction on ϕ. Let $(P;Q)$ be some such circular pair, and suppose for those circular pairs with lower values of ϕ we have deduced the desired sign of the minor (desired means corresponding to which circular pairs are connected and which are not). Assume, without loss of generality, that P is not a consecutive sequence of vertices. Let $p_c + 1 \notin P$ be a vertex with minimal $d(p_1, p_c+1)$. Consider the following relation:

	Δ	a	b	c	d
(**)(12)	$(p_1,\dots,p_c,p_c+1,p_{c+1},\dots p_m);(q_1,\dots,q_m)$	p_1	p_c+1	p_m	q_1

We have $(P;Q) = \Delta^{b,\emptyset}$. Note that all other terms in the relation have lower values of ϕ. We claim that $\Delta^{ac,d} > 0$. Assume the contrary. Because of the induction hypothesis, we have that if $\Delta^{ac,d} \le 0$, then $\Delta^{ac,d} = 0$. Furthermore, it can be readily checked that if this is the case $(P;Q)$ must be a non-solid circular pair of the form:

- $(k+1,\dots,2k+1)^p;(k,\dots,1)$
- $(k+2,\dots,2k+1,1)^p;(k+1,\dots,2)$

- $(1, \ldots, k+1)^p, (2k+1, \ldots, k+2)$

However, we have shown the positivity of such minors. So we must have $\Delta^{ac,d} > 0$. Also, by the induction hypothesis, we have deduced that each term on the right hand side is non-negative. Suppose in (12) the right hand side evaluates to 0. This implies $\Delta^{b,\emptyset} = 0$. This would mean that any circular planar graph with circular pairs in S_1 connected necessarily does not have $(P; Q)$ connected. However, the equivalence class described in case 4 above gives us a contradiction. Thus, the right hand side of 11 must be positive. Therefore, we have that $\Delta^{b,\emptyset} > 0$. By induction, we have therefore deduced the desired signs of all non-solid circular pairs, and we are done. $\square$

Theorem 4.11. *Conjecture 4.2 holds for odd n and $r = 1, 2$.*

Proof. We must find r-positivity tests for cases $1, 2, 3, 4$ presented above. The proof of case 4 is given in Lemma 4.8. For cases $1, 2, 3$, we partition $\mathcal{D}$ into positive and 0 parts. The proof follows readily from a slight modification of the argument of [ALT2, Lemma 4.1.8]. $\square$

5 Conclusions and Discussion

In this paper, we have addressed a novel question of how many Y-Δ equivalences are needed to relate equivalent critical circular planar graphs. Critical graphs, being the "simplest" elements of their equivalence classes, have potential for improved modeling of real networks. In the future, perhaps our bound can be made cubic in n. We have also discussed positivity tests, proving certain cases of a conjecture by Kenyon and Wilson. An inherently algorithmic topic, positivity tests may have interesting implications in analyzing network efficiency.

For future research, it would be interesting to explore the analogies and connections between the space of electrical networks and the totally nonnegative Grassmannian [Pos], which have related applications in physics. Building off these analogies, one can better characterize equivalence classes of EP_n and study certain topological properties of the poset (e.g. lexicographic shellability) [ALT].

6 Acknowledgements

I would like to thank Carl Lian for mentoring my research and providing useful discussions and input concerning the direction of this project. I am also grateful to the PRIMES program and MIT for making this research possible. I would also like to thank Dr. Tanya Khovanova for helping to edit my paper.

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