Combinatorial Games of No Strategy

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Abstract

In this paper, we study a particular class of combinatorial game motivated by previous research conducted by Professor James Propp, called *Games of No Strategy*, or games whose winners are predetermined [10]. Finding the number of ways to play such games often leads to new combinatorial sequences and involves methods from analysis, number theory, and other fields. For the game *Planted Brussel Sprouts*, a variation on the well-known game Sprouts, we find a new proof that the number of ways to play is equal to the number of spanning trees on n vertices, and for *Mozes' Game of Numbers*, a game studied for its interesting connections with other fields, we use prior work by Alon to calculate the number of ways to play the game for a certain case. Finally, in the game *Binary Fusion*, we show through both algebraic and combinatorial proofs that the number of ways to play generates Catalan's triangle.

1 Introduction

Today, game theory is a powerful tool in economics, computer science, biology, and even philosophy [13]. The models it provides allow for mathematics to be applied to great effect. Combinatorial game theory is a subset of game theory which focuses on games whose rules are known and involve no luck. Besides being of great importance to theoretical computer science, it allows for more bridges to be built between pure and applied mathematics.

A **game of no strategy** is a combinatorial game that has a predetermined winner based on the order of play [10]. The winner is defined as the last player to be able to make a move. These games naturally arise from considerations involving processes, as they are interesting for properties not related to strategy. Study of such games may lead to a better understanding of the combinatorial ideas involved, and even topics beyond.

In our research, we study the number of ways to play various games of no strategy, as well as their end states. In Section 2 of this paper, we give examples of fixed length games, which comprise a large class of no-strategy games. In the remaining sections, we study specific games in greater depth. Section 3 concerns a variation of the famous game Sprouts, for which we give a new proof of the number of ways to play, and Section 4 revisits Mozes' Game of Numbers, calculating the number of ways to play on certain starting configurations. Finally, Section 5 studies a simple game involving zeroes and ones which has unexpected relations with Catalan's triangle.

2 Fixed Length Games

In this section we lay out examples of no strategy games, particularly fixed length games. If a game will terminate from its starting position in a fixed number of moves, by parity considerations it is clear that it is a no strategy game.

The winning positions, end states, and proofs of no strategy for these games are not difficult to derive. Here, we calculate the number of ways to play them.

1. We start with a pile of n indistinguishable chips. Each move, a player divides the pile into two piles. The first player who cannot move loses.

   First Terms: Suppose A_n is the number of ways to play on a pile of n chips. Then $A_1 = A_2 = A_3 = 1, A_4 = 2, A_5 = 4, A_6 = 11, A_7 = 33$.

   Corresponding OEIS Sequence: A005470.

   Other Interpretations for this Sequence: The number of unlabeled planar simple graphs with n vertices.

2. We start with a pile of n distinguishable chips. Each move, a player divides the pile into two piles. The first player who cannot move loses.

   First Terms: Suppose A_n is the number of ways to play on a pile of n chips. Then $A_1 = A_2 = 1, A_3 = 3, A_4 = 18, A_5 = 180, A_6 = 2700$.

   Recursion: $A_n = \frac{1}{2}\sum_{i=1}^{n-1}\binom{n}{i}\binom{n-2}{i-1}A_iA_{n-i}$.

   Closed Form: $A_n = \frac{n!\cdot(n-1)!}{2^{n-1}}$.

   Corresponding OEIS Sequence: A006472

3. Start with numbers $1, \ldots, n$ written on the game board. Each move, a player removes two numbers and replaces them with their sum.

   First Terms: Suppose A_n is the number of ways to play with n numbers. Then $A_1 = A_2 = 1, A_3 = 3, A_4 = 18, A_5 = 180, A_6 = 2700$.

   Closed Form: $A_n = \dfrac{n!\cdot(n-1)!}{2^{n-1}}$.

   Corresponding OEIS Sequence: A006472.

   Another Interpretation for this Sequence: By going backwards, this game is equivalent to the previous game. Thus, $\frac{1}{2}\sum_{i=1}^{n-1}\binom{n}{i}\binom{n-2}{i-1}A_iA_{n-i} = \frac{n!\cdot(n-1)!}{2^{n-1}}$.

4. **Chocolate Break:** Given a rectangular $m \times n$ chocolate bar, a player wants to break it into the 1×1 squares. The player is only allowed to break it along the grid lines and cannot break two or more pieces at once.

   First Terms: Suppose $A_{m,n}$ is the number of ways to play on an $m \times n$ chocolate bar. Then the values $A_{m,n}$ takes in increasing order are

   $1, 2, 4, 6, 24, 56, 120, 720, 1712, 5040, 9408, 40320, 92800, 362880, 3628800, 4948992.$

   Recursion: $A_{m,n} = \sum_{i=1}^{m-1} \binom{mn-2}{in-1} A_{i,n} A_{m-i,n} + \sum_{i=1}^{n-1} \binom{mn-2}{im-1} A_{i,m} A_{n-i,m}$.

   Corresponding OEIS Sequences: A261746, A261964, A261747, A257281

   We researched this game in depth with Tanya Khovanova in our paper [4].

Now we consider the three other major games we studied in our research.

3 Planted Brussel Sprouts

3.1 Statement of Game

The following game is a new variation of the classic game Sprouts [3].

On a circle, n notches are drawn. On each move, a player connects two distinct notch ends with a curve within the circle and draws another notch on that curve. Aside from the beginning ones, notches are two-sided; i.e. they can be connected to notches on both sides of the newly-drawn curves. Once there are no more available moves, the game ends.

By induction, it is easy to see that this game ends in $n-1$ moves. It is also possible to assign a planar graph to the game and use the Euler characteristic $V - E + F = 2$ to finish the proof. Indeed, if the game ends in m moves, we have $V = m + n$, $E = 2m + n$, and $F = n + 1$, giving the desired result.

3.2 Number of Ways to Play

The following theorem was proven by [10]; we give another proof.

Theorem 1. *The number of ways to play with n notches is n^{n-2}.*

Proof. Suppose the number of ways to play on a circle starting with n notches is x_n. Note that if the first drawn curve has length i, it divides the game into two sub-games, one with i notches and the other with $n-i$ notches. This leads to the recursion $x_n = \frac{n}{2}\sum_{i=1}^{n-1}\binom{n-2}{i-1}x_i x_{n-i}$.

This gives the values $x_1 = 1, x_2 = 1, x_3 = 3, x_4 = 16, x_5 = 125, x_6 = 2196$. Recall that Cayley's formula for the number of spanning trees on n vertices is also n^{n-2}. We need to show that the number of spanning trees and the number of ways to play follow the same recursion.

Consider all spanning trees on vertices $1, 2, \ldots, n$. Since each edge exists in the same number of spanning trees and there are $n-1$ edges in each tree, exactly $\frac{n-1}{\binom{n}{2}} = \frac{2}{n}$ of them have an edge between vertices 1 and 2. If 1 and 2 are connected by an edge, then if we erase this edge we know that every other vertex needs to be in exactly one of the two subtrees. If there are $i-1$ other vertices in the subtree with vertex 1, then there are $n-i-1$ vertices in the subtree with the vertex 2, so we have the recursion $T_n = \frac{n}{2}\sum_{i=1}^{n-1}\binom{n-2}{i-1}T_i T_{n-i}$, as desired. $\square$

4 Mozes' Game of Numbers

4.1 Statement of the Game

Problem 3 of the 1986 IMO is as follows [8]:

To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y$, $-y$, $z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

It has been shown that that the number of moves for the above puzzle is fixed for any n-gon, allowing this problem to be interpreted as a no-strategy game [9]. This game allows for even further generalizations, which have been extensively studied in References [6, 5], but we could not find any research on the number of moves or the number of ways to play. We give these results for certain cases of the generalized game.

4.2 The game on a 2-gon

Given numbers (a, b), an operation on negative number a replaces a, b with $-a$, $b + 2a$.

Theorem 2. *If a, $k > 0$ and the starting position is $(-a, a + k)$, then the game will take $\left\lfloor \dfrac{a+k-1}{k} \right\rfloor$ moves.*

Proof. We use induction on a. For $a = 1, 2, \ldots, k$, clearly the game will take one move as desired. Assume the result for all $a = 1, 2, \ldots, ik$ for some positive integer i. Then for $a = ik + r$ for $1 \leq r \leq k$, one move will result in $(ik + r, (-i + 1)k - r)$, which by the inductive hypothesis will take i moves. Thus when $a = ik + r$, the game will take $i + 1$ moves as desired, proving the inductive step. $\square$

4.3 Number of ways to play

Theorem 3. *Beginning with numbers* $-a$, $2k+1$, $-2k+a$, 0, 0, 0, $\ldots$ *on an* $(m+2)-$*gon with* k, m, $a>0$, *there are* $\binom{2mk}{ma}$ *ways to play.*

Proof. Consider the set of all arc-sums of numbers; note that they are preserved other than the one directly being changed by a move, which is decreased by 1 [2]. The process ends when there are no negative arc sums. At first, the negative arc sums here consist of m arcs summing to $-a$'s, m arcs summing to $-2k+a$'s, and one arc of $-2k$. Then, by [2], the process will take a total of $2k(m+1)$ moves. Note that for the first k moves, one can make a move on one of the two negative numbers, until they meet on the move number $k+1$. Then move number $k+1$ splits them apart, and the process continues. Thus out of the $2km$ moves which are not move number $n(k+1)$ for some positive integer n, we need to choose ma of them to eliminate the negative arc sums coming from the original $-a$, while the rest eliminate the negative arc sums coming from the original $-2k+a$. Then there are $\binom{2mk}{ma}$ ways to choose which moves occur on which side, as desired. $\square$

5 Binary Fusion

5.1 Statement of the Game

The following game is taken from *Mathematical Circles: Russian Experience* [7].

There are m zeroes and n ones on a board where m and n are positive integers. On a turn, a player erases any two numbers and replaces them with a zero if they are the same number and a one if they are different numbers. There are two players. The first player wins if the final number is a zero, and the second player wins if the final number is a one [7].

Clearly this game ends in $m + n - 1$ moves and, since the sum of all the numbers taken (mod 2) is invariant, it is a no strategy game.

5.2 Number of Ways to Play

In calculating the number of ways to play, we assume that zeroes are indistinguishable from each other, as well as ones. Furthermore, two moves are indistinguishable iff they produce the same result, so combining two zeroes is equivalent to combining a zero and a one.

Define $f(m, n)$ to be the number of ways to play this game with m zeroes and n ones. Let us set $f(0, 0) = 1$.

We get the following table for the values of $f(m, n)$:

$f(m,n)$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	1	1	1	1	2	2	5	5	14	14
1	1	1	2	2	5	5	14	14	42	42
2	1	1	3	3	9	9	28	28	90	90
3	1	1	4	4	14	14	48	48	165	165
4	1	1	5	5	20	20	75	75	275	275
5	1	1	6	6	27	27	110	110	429	429
6	1	1	7	7	35	35	154	154	637	637
7	1	1	8	8	44	44	208	208	910	910
8	1	1	9	9	54	54	273	273	1256	1256

Table 1: Values of $f(m, n)$

5.3 General Recursion

We now derive a recursion for $f(m, n)$ with $m \geq 1$, $n \geq 2$. The first move can combine two zeroes, leaving us with $f(m - 1, n)$ ways left to play. If we combine a zero and a one, we have the same result, so this generates no new ways to play. If we combine two ones, we get

$f(m+1, n-2)$ ways left to play. Thus, we have $f(m,n) = f(m-1,n) + f(m+1,n-2)$ for all $m, n \geq 2$.

5.4 General formula

Theorem 4.

$$f(i,2k) = f(i,2k+1) = \frac{(i+1)(i+k+2)(i+k+3)\cdots(i+2k)}{k!} = \frac{i+1}{i+k+1}\binom{i+2k}{k}.$$

Lemma 5. $f(i,2k) = f(i,2k+1)$

Proof. Since $f(0,i) = f(1,i)$, by the recursion it is clear that $f(i,2k) = f(i,2k+1)$. □

Note: Another way of seeing this is by noting that the parity of the number of ones is invariant. Thus for $m = i, n = 2k+1$, the set of the possible first $i+2k-1$ operations will be the same as the set of the possible total operations for $m = i, n = 2k$. Then a zero and one are left, and there is clearly precisely one way to combine them, so we're done.

Proof of Theorem 4. We prove the result for $f(i,2k)$, which is sufficient because $f(i,2k) = f(i,2k+1)$. We proceed by induction on k. For $k = 1$ there are $i+1$ total steps, i of which are eliminating one zero and one of which is combining two ones. We can choose which one of these to be the one combining two ones, so there are $i+1$ ways to play as desired.

Assume the statement for some $k \geq 1$; we now prove the statement for $k+1$. To do this, we use a second induction on i. For $i = 0$, from the recursion and the induction hypothesis we have

$$f(i,2k) = f(0,2k) = f(1,2k-2) = \frac{2}{2k}\binom{1+2k}{k} = \frac{1}{k+1}\binom{2k}{k},$$

as desired. Now suppose the statement holds for some $i \geq 0$. Then we have:

$$\begin{aligned}
f(i, 2k+2) &= f(i-1, 2k+2) + f(i+1, 2k) \\
&= \frac{i}{i+k+1}\binom{i+2k+1}{k+1} + \frac{i+2}{i+k+2}\binom{i+2k+1}{k} \\
&= \frac{(i+2k+1)(i+2k)\cdots(i+k+2)(i+k+2)i}{(k+1)!} + \frac{(i+2k+1)(i+2k)\cdots(i+k+2))(i+2)}{k!} \\
&= \frac{(i+2k+1)(i+2k)\cdots(i+k+2)(i(i+k+2)+(i+2)(k+1))}{(i+k+2)(k+1)!} \\
&= \frac{(i+2k+1)(i+2k)\cdots(i+k+2)(i^2+2ik+3i+2k+2)}{(i+k+2)(k+1)!} \\
&= \frac{(i+2k+1)(i+2k)\cdots(i+k+2)(i+2k+2)(i+1)}{(i+k+2)(k+1)!} \\
&= \frac{i+1}{i+k+2}\binom{i+2k+2}{k+1},
\end{aligned} \tag{1}$$

as desired. The completes the induction on i, which in turn completes the induction on k, completing the proof.

□

5.5 Catalan's triangle

The Catalan numbers are defined as $C_n = \frac{1}{n+1}\binom{2n}{n}$, and are famous for having an enormous number of combinatorial interpretations. These are generalized by the Catalan triangle, whose entries are given by $C_{n,k} = \dfrac{(n+k)!(n-k+1)}{k!(n+1)!}$ [12]. By Theorem 4, we have $C_{n,k} = f(n-k, 2k) = f(n-k, 2k+1)$, so the number of ways to play this game generates the Catalan triangle.

The combinatorial definition of the Catalan triangle is as follows: The numbers $C(n,k)$ give the number of strings consisting of n X's and k Y's such that no initial segment of the string has more Y's than X's. [12] Here we give a bijective proof of the connection we proved algebraically.

Theorem 6. $C_{n,k} = f(n-k, 2k)$.

Proof. Consider $f(n-k, 2k)$. Denote every move removing a zero to be an X and every move removing a pair of ones and adding a zero to be a Y. After the game is over, we know there must be one zero remaining, so add a move that removes it as an X. Now consider all possible ways to play the game and their respective sequences backwards. We know there must be precisely k Y's, which means there must be a total of $n-k+k=n$ X's. If at some point in such a sequence (going backwards), there are more Y's than X's, then let there be c X's and c' Y's with $c' > c$. Then looking at the sequence forwards, there must be $n-c$ X's and $k-c'$ Y's played to get to the corresponding position. But then at that point, there have been $n-k+k-c' = n-c'$ zeroes total at that point or before, while $n-c$ zeroes have been removed. Since $n-c > n-c'$, this is impossible. Conversely, if no initial string starting backwards has more Y's than X's, then there clearly will be enough zeroes and pairs of ones to remove at each point, so each sequence corresponds to a playing of the game. Thus it is clear we have a bijection. □

Corollary 7. $C_n = f(0, 2n) = f(1, 2n-2)$.

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