Bounded Tiling-Harmonic Functions on the Integer Lattice

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Abstract

Tiling-harmonic functions are a class of functions on square tilings that minimize a specific energy. These functions may provide a useful tool in studying square Sierpiński carpets. In this paper we show two new Maximum Modulus Principles for these functions, prove Harnack's Inequality, and give a proof that the set of tiling-harmonic functions is closed. One of these Maximum Modulus Principles is used to show that bounded infinite tiling-harmonic functions must have arbitrarily long constant lines. Additionally, we give three sufficient conditions for tiling-harmonic functions to be constant. Finally, we explore comparisons between tiling and graph-harmonic functions, especially in regards to oscillating boundary values.

1 Introduction

Tiling-harmonic functions are a class of discrete functions on the vertices of square tilings. They are defined to minimize a specific energy given the boundary values of the tiling. The motivation for defining and studying tiling-harmonic functions comes from a paper by M. Bork and S. Merenkov [2], which was breakthrough in understanding the quasisymmetric rigidity of non-round carpets. Tiling-harmonic functions are a simplified version of square Sierpiński carpets, defined in the spirit of the transboundary modulus introduced by O. Schramm in [4]. Sierpiński carpets are a type of fractal important in measure theory and topology; they are one way of generalizing the Cantor set to two dimensions. Conjecture 2 is directly relevant to Sierpiński carpets — proving it would simplify several of the more involved arguments of Bonk and Merenkov's paper.

Tiling-harmonic functions are also interesting combinatorial objects in their own right, especially given their similarities to graph-harmonic functions, the discrete analogues of harmonic functions. They share many of the same properties, which we have proved and show in this paper. An introduction to graph-harmonic functions can be found in [3]. Doyle and Snell prove a number of properties of graph-harmonic functions, including a Maximum Modulus Principle, but unfortunately their method relies heavily on the Laplacian, for which there is no equivalent for tiling-harmonic functions. We thus use combinatorial methods and inequalities to build our own tools in this paper.

A set of boundary values which gives a nontrivial difference between graph-harmonic and tiling-harmonic functions is given in [1] and reproduced in this paper (see Figure 23). The Illinois Geometry Lab project also found an algorithm to numerically compute a tiling-harmonic function given a set of boundary values.

This paper contains results concerning many properties of tiling-harmonic functions, namely two Maximum Modulus Principles, the closure of the set of harmonic functions, Harnack's Inequality, three sufficient conditions for a tiling-harmonic function to be constant, and a structural feature of bounded infinite tiling-harmonic functions.

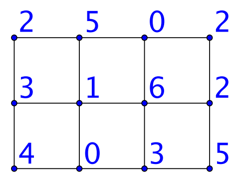
Section 2 defines tiling-harmonic and graph-harmonic functions; this provides an introduction and background to the subject. Our results are contained in Section 3 and Section 4, while Section 5 discusses the differences between tiling and graph-harmonic functions through a series of images. Lastly, sections 6 and 7 look forward to the future of this project.

2 Definitions

Definition 1. A *square tiling* is defined broadly as a connected set of squares in the plane whose interiors are disjoint and whose edges are parallel to the coordinate axes.

Tiling-harmonic functions are a certain class of functions on the vertices of square tilings. For our purposes, we shall only work with connected subsets of the regular square lattice $\mathbb{Z}^2$. An example of a function on a square tiling is shown in Figure 1.

Figure 1: Function on a Square Tiling



Definition 2. A tiling S is a *subtiling* of a tiling T if the set of S's squares is a subset of the set of T's.

Definition 3. The *oscillation* $\operatorname{osc}(u,t)$ of a function on a square t is the difference between the maximum and minimum values on that square.

Definition 4. The *energy* $E(u)$ of a function on a tiling is the sum over all squares t in that tiling of the square of the oscillation on t, i.e.,

$$E(u) = \sum_t (\operatorname{osc}(u,t))^2.$$

Example 2.1. The function in Figure 2 has the energy

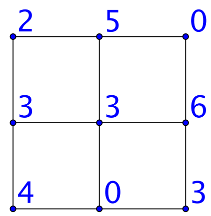
$$E(u) = (5-2)^2 + (6-0)^2 + (4-0)^2 + (6-0)^2 = 97.$$

Definition 5. A function on a finite square tiling is called *tiling-harmonic* if it minimizes the energy among all functions on that tiling with the same boundary values.

A function on an infinite tiling is called *tiling-harmonic* if it is tiling-harmonic on all finite subtilings.

Remark. Given a tiling and a set of boundary values, tiling-harmonic functions are not necessarily unique.

Figure 2: An Example Function



Example 2.2. The function in Figure 2 is tiling-harmonic because the only non-boundary value, the 3 in the center, is not a maximum or minimum for any of the four squares. Thus the oscillations are minimized, and therefore the energy is minimized.

Note also that if we replaced the 3 in the center by a 4 (or any value between 3 and 4 — tiling-harmonic functions can take any real value), it would still be tiling-harmonic for the same reason, so this tiling-harmonic function is not unique.

Theorem 1. *The function $f(x, y) = y$ is tiling-harmonic.*

Figure 3: The Height Function

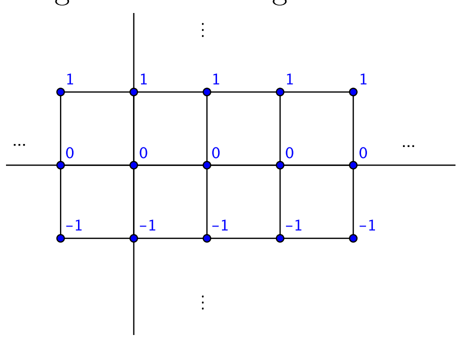
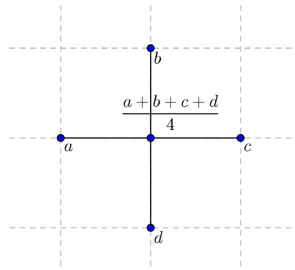


Figure 3 shows a portion of the height function. A proof of Theorem 1 can be found in [1].

Definition 6. A function on a square tiling is called *graph-harmonic* if the value at each vertex is the average of the values of its neighbors.

The defining property of graph-harmonic functions is shown in Figure 4. This is the discrete analogue to the harmonic functions typically seen in complex analysis. Many properties of graph-harmonic functions are known. For example, the set of graph-harmonic functions on a given square tiling forms a vector space. Graph-harmonic functions are unique — given a tiling and its boundary values, there is a unique graph-harmonic function satisfying them.

Figure 4: The Graph-Harmonic Function Property



3 Properties of Tiling-Harmonic Functions

In this section we present the first half of the results of the project.

We start with a useful general property of tiling-harmonic functions. This will first require a definition of scalar addition and multipication of tiling-harmonic functions.

Definition 7. If f is a function on a square tiling, then $g = af$ is the function that results from multiplying each of the values of f by a, and $h = f + b$ is the function that results from adding b to the each of the values of f.

Lemma 2. *If f is a tiling-harmonic function, then $af + b$ is also tiling-harmonic.*

Proof. Because energy is defined in terms of oscillations, scaling and shifting a function make no difference to whether the energy is minimal. □

The first theorem of this section is a property that says no tiling-harmonic function can have a local maximum.

Theorem 3 (Maximum Modulus Principle for Values)**.** *On an $m \times n$ rectangular grid with $m, n \geq 4$, if the maximum value of a tiling-harmonic function occurs on the interior, then the entire set of interior values is constant.*

Proof. We shall prove here the equivalent (in fact slightly stronger) statement that any nonnegative function with an interior zero must have all its interior values zero and its boundary squares each must contain at least one boundary zero. This statement is equivalent to the theorem by Lemma 2 — we multiply our general function by -1, then shift it so that its minimum is 0 to get the nonnegative function we will be working with here.

Take any function with an interior zero that does not satisfy all the above conditions; we shall prove it is not tiling-harmonic.

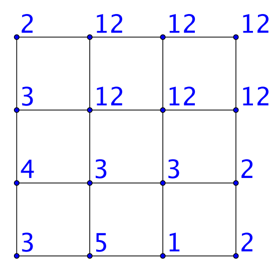
Consider an interior square that has both a zero and a nonzero value, or take a boundary square with no zeroes on the boundary portion. One of these must be possible because the function does not satisfy the above conditions.

Increase all interior zeroes by ϵ. Say our chosen square S has maximum value M. Then the energy of square S increases by $(M+\epsilon)^2 - M^2 = \epsilon^2 - 2M\epsilon$. Other squares increase either by ϵ^2 or $\epsilon^2 - 2k\epsilon$ for some k particular to that square, resulting in a total increase of $a\epsilon^2 - b\epsilon$ for some $a, b > 0$. But for $0 < \epsilon < \frac{b}{a}$ this increase of energy is negative, and hence the new function has a lower energy, thus our original function was not tiling-harmonic. □

There is an analogous theorem for graph-harmonic functions. Additionally, by multiplying the function by -1 (which retains its tiling-harmonicity), we have a Minimum Modulus Principle for Values as well.

Example 3.1. The function in Figure 5 cannot be tiling-harmonic because the maximum, 12, occurs in the interior but the entire interior is not constant.

Figure 5: This function is not tiling-harmonic.



Beginning after Figure 5, we will remove the grid lines from most of our figures of tiling-harmonic functions to unclutter the pictures.

The following theorem states that the set of tiling-harmonic functions is closed.

Theorem 4. *The limit of a sequence of tiling-harmonic functions is itself tiling-harmonic.*

Proof. First, we define how we take a limit and which sequences converge. A sequence of tiling-harmonic functions converges if and only if all of the functions are on the same square tiling, and the sequence of values on each vertex of the tiling converges. The limit is the function with value on each vertex equal to the limit of that vertex's sequence of values.

Since a function on an infinite tiling is called tiling-harmonic if and only if the restriction of the function to every finite subtiling is tiling-harmonic, we only need to consider the case where the tiling is finite. Without loss of generality, say that the values on the vertices for all the functions are between 0 and 1 — we can assume this by Lemma 2, as there are only a finite number of values. For the sake of contradiction, assume the limit L of the tiling-harmonic functions is not itself tiling-harmonic, i.e., there is some function G with the same boundary values as L that has a lower energy. We denote the energy of a function F

on a tiling to be $E(F)$. Additionally, we denote the difference $E(L) - E(G)$ as $E_0 > 0$. Let n be the number of boundary squares of our square tiling, and m be the total number of squares of our tiling.

Now take a tiling-harmonic function T in the sequence such that T differs from L by at most ϵ on all vertices. Then consider the function H defined to be identical to T on the boundary values and to G on the interior values. The oscillation of each of the n boundary squares of H is at most 2ϵ greater than the corresponding oscillation in G, while the interior squares have the same oscillation as those in G. Increasing the oscillation by 2ϵ will increase the energy the most when the original oscillation was as large as possible (that is, equal to 1), and thus

$$\begin{aligned} E(H) &\leq E(G) + n((1+2\epsilon)^2 - 1^2)) \\ &= E(G) + n(2\epsilon + \epsilon^2). \end{aligned}$$

Now consider the energy of T. The oscillation of each of the m squares of T is at most 2ϵ smaller than the oscillation of the corresponding square in L, and again, the difference in energy is maximized when the original oscillations (in L) were all equal to 1, thus

$$\begin{aligned} E(T) &\geq E(L) - m(1^2 - (1-2\epsilon)^2) \\ &= E(L) - m(2\epsilon - \epsilon^2) \\ &= E(G) + E_0 - m(2\epsilon - \epsilon^2) \\ &\geq E(H) + E_0 - n(2\epsilon + \epsilon^2) - m(2\epsilon - \epsilon^2). \end{aligned}$$

We choose $\epsilon < \frac{E_0}{2n+2m}$ so that

$$n(2\epsilon + \epsilon^2) + m(2\epsilon - \epsilon^2) < E_0 - (m-n)\epsilon^2 \leq E_0,$$

since $m - n$ is the number of interior squares and is thus nonnegative. So this choice of ϵ gives

$$E(T) > E(H),$$

but T is tiling-harmonic and T and H have the same boundary values, giving us a contradiction. Hence our assumption was false and L is tiling-harmonic. □

The set of graph-harmonic functions on a given set of boundary values is also closed. Unlike graph-harmonic functions, however, tiling-harmonic functions do not form a vector space.

The following theorem and corollary concern Harnack's Inequality for Tiling-Harmonic

Functions. This was originally given as a conjecture by Merenkov.

Theorem 5 (Harnack's Inequality for Tiling-Harmonic Functions)**.** *On a nonnegative, bounded infinite tiling-harmonic function, the ratio between any two points a distance d apart is bounded by a function of d.*

Note that the distance metric we use is $d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$.

Proof. In this proof, we will first show that assuming Harnack's Inequality is false leads to a point Q that limits to 0. We then use Q to find a square S_R with exactly three points that limit to 0. Finally, we use an argument very similar to that in the proof of Theorem 3 to show that the existence of S_R leads to a contradiction.

Fix d, and assume that Harnack's Inequality is false. Thus there exists a sequence of values a_n such that $\lim_{n\to\infty} a_n$ is unbounded, where for each a_n there exists a tiling-harmonic function F_n with two points a distance d apart that have ratio a_n. There are only a finite number of places to position the point corresponding to the denominator of the ratio relative to the point corresponding to the numerator, so one of these orientations must occur infinitely many times. Consider the infinite subsequence $\{F'_n\}$ of $\{F_n\}$ that consists of only the functions with this orientation, and shift the points of each F'_n so that each has the two points it uses for its ratio in the same locations. Call P the location of the point corresponding to the numerator, and Q the location of the point corresponding to the denominator.

Consider a square S with side length $s > d$ containing P and Q. Scale each function F'_n so that the boundary of S has maximum 1. Now we will show some subsequence $\{F''_n\}$ of $\{F'_n\}$ will have a limit L that exists, with maximum 1. First, some boundary point will be the maximum of the function — and thus equal to 1 — an infinite number of times, so consider that infinite subsequence. Then choose some other point on the function: its values are bounded in the range $[0, 1]$ so there must exist an accumulation point; consider the subsequence (of the previous subsequence) that limits to this accumulation point. Repeat for each point on the boundary and all interior points, and finally the resulting subsequence $\{F''_n\}$ will have a limit L.

The limit L is tiling-harmonic by Theorem 4, and its values are bounded, so $\{F''_n(Q)\}$ must limit to 0, and thus by Theorem 3, the entire interior must limit to 0. Theorem 3 additionally says that any boundary square with a nonzero value that is not a corner must have all three other values limit to zero; since there exists at least one nonzero boundary value (i.e., the maximum), at least one such square with exactly three zeroes must exist. Let R be the point which contains the nonzero value of one of these squares S_R.

Now consider, for each function F''_n, the $(s + 2) \times (s + 2)$ square immediately surrounding our $s \times s$ square. Let the maximum of this square for the function F''_n be μ_n. Then if $\lim_{n\to\infty} \mu_n$

exists, each point of the function has its values bounded, so we can take a subsequence which limits. The interior of this limit will be exactly L, so there will be both a 0 and a 1 in the interior of that limit — that is, the interior is not constant — and this contradicts Theorem 3. So $\lim_{n\to\infty} \mu_n$ must be unbounded. Then renormalize each function F_n'' so that the maximum of the $(s+2) \times (s+2)$ square equals 1. Let this renormalized sequence be $\{F_n'''\}$, and let the new values on the point R be r_n, as in Figure 6. Clearly, after renormalizing, $\lim_{n\to\infty} r_n = 0$. But if we let T_1, T_2, T_3 be the three points other than R in S_R and $c_{i,n}$ be the set of values on T_i, then $\lim_{n\to\infty} \frac{r_n}{c_{i,n}}$ approaches infinity for each i, since renormalizing does not affect ratios.

Figure 6: L (left) and F_k''' (right)

$$\begin{array}{rcccc} & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ R-> & \frac{1}{2} & 0 & 0 & 0 \\ & 0 & 0 & \frac{3}{4} & 0 \end{array} \qquad \begin{array}{cccccc} * & * & * & * & * & * \\ * & & & & & * \\ * & c_{1,k} & c_{2,k} & & & * \\ * & r_k & c_{3,k} & & & * \\ * & & & & & * \\ * & * & * & * & * & * \end{array}$$

Consider some function F_k''' such that $\frac{r_k}{c_{i,k}} > t$ for every $c_{i,k}$, where $t = 2(s+1)^2 + 1$. Then consider increasing every interior value that is less than $\frac{1}{t} r_k$ to the value $\frac{2}{t} r_k$. We will show that this is a decrease in energy.

We look at the change in energy one square at a time. If the energy of a square increases, then $\frac{2}{t} r_k$ must be its new maximum, so the increase is by at most $(\frac{2}{t} r_k)^2 - 0^2 = \frac{4r_k^2}{t^2}$. On the other hand, the energy of at least one square, namely S_R, decreases. The maximum of that square is r_k, so the energy decreases by at least $(r_k - \frac{1}{t} r_k)^2 - (r_k - \frac{2}{t} r_k)^2 = \frac{2r_k^2}{t} - \frac{3r_k^2}{t^2}$. When all the energies are summed up, we get a decrease of at least $\frac{2r_k^2}{t} - \frac{4r_k^2(s+1)^2}{t^2}$, which is positive since $t > 2(s+1)^2$. So our function F_k''' is not tiling-harmonic, which is a contradiction, and thus Harnack's Inequality is true. □

Corollary 6. *On a nonnegative, bounded infinite tiling-harmonic function, the ratio between any two points a distance d apart is bounded by an* ***exponential*** *function of d.*

Proof. Let our bound for values a distance of 1 apart be α. Then by choosing a path of n points with the distance between each neighboring pair equal to 1, the bound for n must be at most α^n. □

Note that whenever we mention Harnack's Inequality in this paper, we mean the analogous version for tiling-harmonic functions unless we specifically say otherwise.

This section's final result is a sufficient condition for a constant tiling-harmonic function.

Definition 8. Given a function f on a finite tiling, for every boundary square, take the range of the boundary values on that square. If the intersection of these ranges is nonempty, we say that f has oscillating boundary values.

Theorem 7. *If a function f has oscillating boundary values, then f is tiling-harmonic only if it is constant on the interior.*

Proof. The boundary values themselves set a minimum energy for a tiling. Take the minimum and maximum of the boundary values on each boundary square — this produces a minimum oscillation for each boundary square. Summing the squares of these oscillations results in a minimum energy for the tiling.

If the boundary values are oscillating, then there exists a value c such that the function f that takes the value c on each of the interior points will not increase the oscillation of any of the boundary squares from their minimum oscillations. Having c on each of the interior points also means that the oscillation of each interior square is equal to 0, so the minimum energy of the tiling, specified by the boundary values, has been realized.

Thus the function f that takes the value c on all interior points is tiling-harmonic, and any function that has a positive oscillation on any interior square (i.e., any function that is not constant) will have a greater energy. □

Example 3.2. The ranges of the boundary values of the boundary squares in Figure 7 all include the number 1, so the only tiling-harmonic functions satisfying this set of boundary conditions have constant interior.

Figure 7: A function with oscillating boundary values

$$\begin{matrix} 2 & 5 & 0 & 2 \\ 1 & 1 & 1 & 0 \\ 4 & 1 & 1 & 5 \\ 1 & 2 & -1 & 0 \end{matrix}$$

Remark. Note that the condition of having oscillating boundary values is sufficient but not necessary, as the function in Figure 8 is tiling-harmonic and constant on the interior, but does not have oscillating boundary values.

Example 3.3.

It is still open whether this condition is necessary for m and n both greater than 4.

Figure 8: A counterexample

0	1	1	0
−1	0	0	−1
−1	0	0	−1
0	1	1	0

4 Considering Oscillations

This section contains the second half of the results of this project. The focus in this section is the oscillations of the squares, rather than the individual values.

The first result is a useful lemma, which will be used in multiple proofs.

Lemma 8. *If a square S has a unique maximum M, then if we call its minimum m and its oscillation o, the square D that is diagonal to S and intersects S at the value M must have a value in it that is at least $2M - m = M + o$. The relative locations of S and D are shown in Figure 9 below.*

Figure 9: The squares S and D

$$\begin{array}{ccccc} & & & - & \\ & & | & \mathbf{D} & | \\ & - & M & - & \\ | & \mathbf{S} & | & & \\ & - & & & \end{array}$$

Proof. To make the notation simpler, we will prove the following equivalent statement: if a square S has a unique maximum M in its top right corner and minimum $m = 0$, then the square above-right must have a value in it that is at least $2M$.

In Figure 10 below, S is the bottom-left square. We also have that at least one of d, f, g is 0, and that $d, g < M$. We will prove this lemma by contradiction. Assume otherwise, that is, in Figure 10, $b, c, e < 2M$. Let $K = \max\{b, c, e\}$.

Figure 10: S and its neighboring squares

$$\begin{array}{ccccc} a & - & b & - & c \\ | & & | & & | \\ d & - & M & - & e \\ | & \mathbf{S} & | & & | \\ f & - & g & - & h \end{array}$$

Decrease M to $M' = \max\{d, g, f, \frac{K}{2}\}$. Note that this is a nonzero decrease by our assumption. We will show that this decreases the energy of the set of four squares, contradicting our assumption and thereby proving our claim.

Since $M' \geq d$, we are not decreasing the minimum of the top-left square, so its energy does not increase. Similarly, $M' \geq g$ so the bottom-right square's energy does not increase. If one of $b, c, e < M'$ then the top-right square's energy also does not increase (and the bottom-left square's energy decreases, so we're done), so assume $M' \leq b, c, e$, i.e., M' is the minimum of the top-right square.

Let E be the original energy with M, and E' be the new energy with M'. Let the four squares be $TR =$ top-right, $TL =$ top-left, $BR =$ bottom-right, and $BL =$ bottom-left $= S$. Then

$$\begin{aligned}
E &= E(TL) + E(BR) + E(TR) + E(BL) \\
&\geq E'(TL) + E'(BR) + E(TR) + E(BL) \\
&= E'(TL) + E'(BR) + E(TR) + M^2 \\
&\geq E'(TL) + E'(BR) + (K - M)^2 + M^2 \\
&> E'(TL) + E'(BR) + (K - M')^2 + M'^2 \\
&= E'(TL) + E'(BR) + E'(TR) + E'(BL) \\
&= E'.
\end{aligned}$$

We have $(K - M)^2 + M^2 > (K - M')^2 + M'^2$ because $f(x) = (K - x)^2 + x^2$ is minimized at $x = \frac{K}{2}$ and is smaller for x closer to $\frac{K}{2}$, and we have $\frac{K}{2} \leq M' < M$.

Thus we have a contradiction: our function is not tiling-harmonic, so our assumption must be false, and thus one of $b, c, e \geq 2M$. □

The next theorem shows that the Maximum Modulus Principle applies not just for the individual values, but for the oscillations as well.

Theorem 9 (Maximum Modulus Principle for Oscillations)**.** *The (not necessarily unique) largest oscillation of a tiling-harmonic function on a finite tiling must occur on its boundary.*

Remark. Note that the analogous "Minimum Modulus Principle for Oscillations" is in fact false; for a counterexample, see Example 3.3.

Proof. We will show that for any square S with oscillation o, one of three options occurs:

1. There exists some other square with a larger oscillation.

2. There exist two squares on direct opposite sides of it each with oscillation o.

3. There exists an infinite ray of squares emanating from it all with oscillation o.

This will imply our theorem — if we consider the set of squares with the largest oscillation, then for each of them either Option 2 or Option 3 will be true. If Option 3 is true for any of them, there will be a square with the largest oscillation on the boundary. Otherwise, Option 2 must be true for all of these squares, but there cannot exist a set of interior squares all of which satisfy Option 2 (as such a set cannot have any corners), so in this case too, there must be a square with largest oscillation on the boundary.

Now let us show that one of the three options is true for every square. Take any square S. For simplicity's sake, we will always take the maximum on our initial square to be 1 and the minimum to be 0. We will also assume that the oscillation of S is maximal among the squares of our tiling, otherwise Option 1 is immediately satisfied.

Say that the maximum $M = 1$ and minimum $m = 0$ of the square are on opposite vertices, and that this maximum and minimum are unique. Then by our lemma, the above-right square must have a value at least 2 and thus an equal or greater oscillation. Similarly, the below-left square must have a value at most -1, so in this case Option 2 is satisfied.

Figure 11: Adjacent maximum and minimum

$$
\begin{array}{ccccccc}
 & & & & 2 & - & 3 \\
 & & & & | & & | \\
 & & 1 & - & 2 & - & \\
 & & | & \mathbf{T} & | & & \\
0 & - & 1 & - & & & \\
| & \mathbf{S} & | & & & & \\
x & - & y & & & &
\end{array}
$$

Now say we have a situation where the maximum and minimum (not necessarily unique) are adjacent to each other on the square S, as in the bottom left corner of Figure 11. Note that if there do not exist a maximum and minimum on adjacent vertices, then there must be a unique maximum and unique minimum on opposite vertices, so this and the previous case exhaust all possibilities. Assume for now that 1 is the unique maximum of S, i.e., that $x, y < 1$ in Figure 11.

Call the *range* of a square the range between its minimum and maximum. For example, the range of S is 0 to 1. The square above S has values 0 and 1 and cannot have an oscillation larger than 1, so it must have range 0 to 1.

The square S has a unique maximum, so the square T above-right must have a value in it that is at least 2. It cannot be greater than 2 since there is a 1 in T already and it cannot have oscillation greater than 1, so there must be a 2 in the square. If it occurs in its bottom right corner, then y is in a square with range 1 to 2, so $y \geq 1$, contradiction. Additionally, the 2 cannot be in the top left corner of T because the square above S has range 0 to 1. Thus the 2 is in the top-right corner. Then the top-left corner of T is in a square with range 1 to 2 and in a square with range 0 to 1, so it must be equal to 1.

We continue in this same fashion. The value in the bottom right corner of T must be less than 2, so 2 is the unique maximum of T. Following the same argument, we get a chain of numbers as in Figure 11 that will continue until we hit the boundary. Notice that whenever we get the numbers n and $n+1$ adjacent to each other, we have a square of oscillation 1 (in fact, we get two such squares). Thus in this case Option 3 is satisfied — we have a ray of squares with the same oscillation as S.

Notice that we assumed $x, y < 1$. If we instead assume $x, y > 0$ (i.e., that 0 is the unique minimum), then the same thing can be done with the minimum instead of the maximum, so the only cases left are when $(x, y) = (1, 0)$ and $(x, y) = (0, 1)$. It is immediate that these two cases satisfy Option 2.

Thus our theorem is proved. The largest oscillation must occur on the boundary. □

We now present two corollaries. The first looks at functions with finite total energy.

Corollary 10. *If a tiling-harmonic function on the integer lattice has finite energy, it is constant.*

Proof. For the sake of contradiction, assume that there is a nonconstant tiling-harmonic function on the full plane with finite energy. Since the energy of the function is finite, there must be either be one square with the largest oscillation among all the squares, or else a finite number of squares tied for the largest oscillation. Call this largest oscillation O. Consider a rectangle that contains all squares with oscillation O in its interior. Then by Theorem 9, there must be a boundary square on this rectangle with oscillation at least O, which, by construction, is a contradiction. Thus our initial assumption was false, and every tiling-harmonic function on the whole plane with finite energy is constant. □

The second corollary tells us a great deal about the structure of a tiling-harmonic function on the integer lattice.

Corollary 11 (Arbitrarily Long Constant Lines)**.** *There exist arbitrarily long (or infinite) lines of equal values in any bounded tiling-harmonic function on the integer lattice.*

Proof. Consider any tiling-harmonic function f in the plane. Assume it is not constant, as a constant function immediately satisfies our property. Consider a $n \times n$ square, with $n \geq 4$, with a nonconstant interior. Note we can choose n arbitrarily large. The nonconstant interior implies that the maximum value M of the square cannot occur on the interior (by Theorem 3), so let the maximum value of the interior be $M - \epsilon$, with $\epsilon > 0$. Note that this maximum value must occur on the closed $(n-2) \times (n-2)$ chain of values that is immediately inside the boundary, also by Theorem 3.

Assume for now that there is a boundary square S that has a unique maximum of M. We will shortly describe when this situation does not exist. Let the minimum of this square be $m \leq M - \epsilon$, as every boundary square has at least one value on the $(n-2) \times (n-2)$ chain. In Figure 12 below, the $n \times n$ chain is represented by $+$'s, and the $(n-2) \times (n-2)$ chain by $*$'s. Then by Lemma 8, the square diagonally adjacent to S which intersects S at the value M must have a maximum value K at least

$$M + (M - m) \geq M + \epsilon$$

and thus the $(n+2) \times (n+2)$ chain of values surrounding our $n \times n$ square has a maximum at least $M + \epsilon$. Note that if it is larger, the difference between the maximum of the $(n+2) \times (n+2)$ chain and the $n \times n$ chain is greater than ϵ, so we in effect get a larger value of ϵ when we repeat the process. Since we can do the same process again and again but cannot continue forever since our function is bounded, at some point our assumption must be false, i.e., there does not exist a boundary square with a unique maximum of M.

Figure 12: A boundary square has unique maximum M

$$\begin{array}{cccccc}
 & K & & & & \\
+ & + & M & + & + & + \\
+ & * & * & m & * & + \\
+ & * & & & * & + \\
+ & * & & & * & + \\
+ & * & * & * & * & + \\
+ & + & + & + & + & +
\end{array}$$

Let us see when that is the case. There must be at least one value of M on the boundary. If that value is a corner of the square, one of the two edge values bordering it must also take the value of M (or else our M is a unique maximum of that corner boundary square). Thus there must always be an edge value of M. The value of M cannot appear in the interior, so if a boundary square that is not a corner has the value of M on one of its boundary vertices, it must take the value M on the other boundary vertex as well. Thus, an edge value of M leads to its neighboring boundary vertices taking the value M as well, and these force their neighbors to be M, and so on, until that entire edge (with the possible exception of the corners) is composed of values of M. Figure 13 is an example of this situation where no boundary square has a unique maximum of M. As we can make n arbitrarily large, and the process keeps producing larger and larger squares, the edge of M's is an arbitrarily long straight line of equal values.

Figure 13: No boundary square has unique maximum M

$$\begin{array}{ccccc} M & M & M & M & + \\ + & * & * & * & M \\ + & * & & * & M \\ + & * & * & * & M \\ + & + & + & + & M \end{array}$$

□

The following theorem approaches infinite tiling-harmonic functions from a different direction, specifying a smaller set of values which are allowed in the range of the function.

Theorem 12 (Limited Range)**.** *Let A be any set of numbers which has at most one accumulation point. Any tiling-harmonic function on the infinite lattice which takes no values other than those in A must be constant.*

We will provide two proofs. The first proof will prove the theorem, while the second will prove a simpler version, where A is finite (and hence has no accumulation points). We include both because the two proofs stem from different ideas and either method of proof may prove useful for an extension of Theorem 12.

Proof. For the sake of contradiction, assume that there exists a nonconstant function which takes no values other than those in the set A. Consider a 4×4 square with nonconstant interior, and, as in the proof of Corollary 11, consider a series of disjoint closed chains around the square — the 6×6 chain immediately around it, the 8×8 chain around that, and so on. Let x be one of the values on the interior of our 4×4 square. Let m_k and M_k be the minimum and maximum, respectively, of each $k \times k$ chain. Then by Theorem 3, we have $x < M_6 < M_8 < M_{10} < \ldots$, and $x > m_6 > m_8 > m_{10} > \ldots$. But since our function is bounded, there are thus at least two accumulation points, one greater than x and another less than x. This is a contradiction, thus such a function must be constant. □

The first proof relied heavily on Theorem 3; the second will be dependent on Lemma 8. For this proof, we will need a few lemmas.

Lemma 13. *Given a tiling-harmonic function on the infinite plane that takes no values other than those in the finite set A, of the four values in each square, there must either be four of* *a single value or two each of two different values. In other words, no square has a unique maximum or unique minimum.*

Proof. We shall prove this by induction. Call the property of one square having no unique maximum or minimum "property p"; we want to show all squares must satisfy property p.

Let the finite set $D = \{x - y | x, y \in A, x - y \geq 0\}$ be the set of positive differences between two values of A. Let n be the largest element of D. Without loss of generality, assume that the smallest element of A is 0 and the largest is n. We shall call a square *valid* if there exists a tiling-harmonic function containing that square. Our base case is to show that all valid squares with oscillation n satisfy p.

Consider a square with oscillation n that does not satisfy p. Then there is either a unique maximum of n or a unique minimum of 0, which means that in an adjoining square there is either a value greater than $2n$ or less than $-n$ by Lemma 8, either case obviously a contradiction. Thus our base case is complete.

Now assume all squares with oscillation greater than k satisfy property p. Consider a square S with oscillation equal to k that does not satisfy p. Call its minimum and maximum i and $i + k$, respectively, and without loss of generality, the value $i + k$ is a unique maximum of S and in the upper right corner.

Figure 14: Square S and its neighbors

$$
\begin{array}{ccccc}
 & - & < i+2k & - & i+2k \\
| & \mathbf{T} & | & \mathbf{V} & | \\
< i+k & - & i+k & - & < i+2k \\
| & \mathbf{S} & | & \mathbf{U} & | \\
< i+k & - & < i+k & - &
\end{array}
$$

Thus by Lemma 8, the square V above and to the right of S, as shown in Figure 14, has a value in it at least $i + 2k$, so V has oscillation at least k. If this value of at least $i + 2k$ were in either the upper left or lower right corner of V, we would get an oscillation of more than k in either the square T above or the square U to the right of our original square, with at least three distinct values inside it, so the $i + 2k$ must be in the upper right corner. Then V has a unique maximum and oscillation k, so we can repeat the same process over and over to get successive squares with oscillations k and upper right corner $i + jk$ for $j = 1, 2, 3, \ldots$. Eventually, for large enough j, $i + jk > n$, which is a contradiction. So thus a square with oscillation k cannot have a unique maximum (or, similarly, a unique minimum), so it must have its maximum and minimum each repeated twice.

This completes the induction. Since D is a finite set, we can induct downwards from n to the smallest element of D. □

Lemma 14. *The only tiling-harmonic functions on the integer lattice with at most 2 distinct values are constant.*

Proof. Call our two values a and b, with $a > b$. For the sake of contradiction, assume there exists a tiling-harmonic function with only the values a and b that is not constant. Then there exists a 4×4 square with both a and b in the interior. But then a is the maximum of this 4×4 square, which is tiling-harmonic yet has a non-constant interior. This is a contradiction to Theorem 3. Thus our assumption was false; no such tiling-harmonic function exists. □

Now onto the main proof.

Proof. Say that we have a pair of adjacent unequal values i and j in our function. Without loss of generality, say they are vertically adjacent, with i above j, as shown in Figure 15.

Figure 15: The values i and j

$$\begin{matrix} i \\ j \end{matrix}$$

Then by our previous work, since any square that contains two distinct values must contain two copies of each, the vertical pair of values to their right must be either i over j or j over i. Continuing in the same manner, every vertical pair of values in these two rows must be either i above j or j above i, as shown in Figure 16.

Figure 16: Two rows of i and j

$$\begin{matrix} i & i & j & j & j & i & j \\ j & j & i & i & i & j & i \end{matrix}$$

Assume for now that j and i each occur at least once in the bottom row. Consider the value under one of the i's in the bottom row, and choose this i such that it has a j to either its left or right. Since this value is in a square with two distinct values, i and j, in it already, this value must equal either i or j. We can continue to use Lemma 13 to find that the row below the bottom row must either be the same as the bottom row, or its direct opposite, with all i's exchanged for j's and vice versa (i.e., the same as the top row), as shown in Figure 17.

This same reasoning can be repeated, so the whole plane is made up solely of i's and j's. By Lemma 14, this function is not tiling-harmonic.

Now consider the case where i and j do not both occur at least once in the bottom row. We already know that j occurs once in this row by our initial assumption, so it must be that

Figure 17: A whole plane of i and j

$$\begin{array}{ccccccc} i & i & j & j & j & i & j \\ j & j & i & i & i & j & i \\ j & j & i & i & i & j & i \end{array}$$

the top row is made up solely of the value i, while the bottom row is made up solely of the value j.

If x is the value below one of the j's, Lemma 13 says that all values in x's row must take the value x. Thus we see that each row in the plane is completely made up of a single value.

Consider the row with the largest value M, choosing one that borders a row with a value smaller than M if multiple rows have value M. This row of course exists since our function only takes a finite number of values. There are two possible scenarios:

Figure 18: A single row of M's

$$\begin{array}{cccc} k & k & k & k \\ M & M & M & M \\ l & l & l & l \end{array}$$

If both the rows above and below have values smaller than M as in Figure 18, decreasing two adjacent values of M to $\max(k, l)$ will decrease the energy.

On the other hand, if one of the two adjacent rows has value M, decreasing two adjacent values of M in our row by ϵ, as in Figure 19, will still result in a decrease of the energy.

Figure 19: Adjacent rows of M's are not tiling-harmonic

$$\begin{array}{cccc} k & k & k & k \\ M & M-\epsilon & M-\epsilon & M \\ M & M & M & M \end{array}$$

The net decrease in energy is

$$(M-k)^2 - (M-\epsilon-k)^2 - 4\epsilon^2 = 2(M-k)\epsilon - 5\epsilon^2 = \epsilon(2(M-k) - 5\epsilon),$$

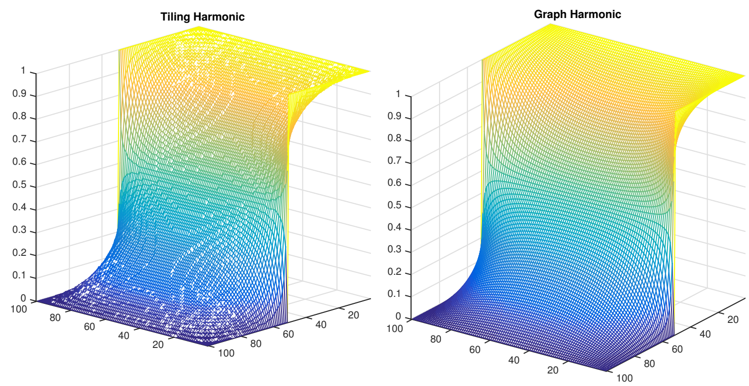
which is positive for $0 < \epsilon < \frac{2(M-k)}{5}$.

So our initial assumption that we had two unequal values, i and j, is wrong. Thus the function must be constant. This concludes our proof. $\square$

5 Comparing Tiling and Graph-Harmonic Functions

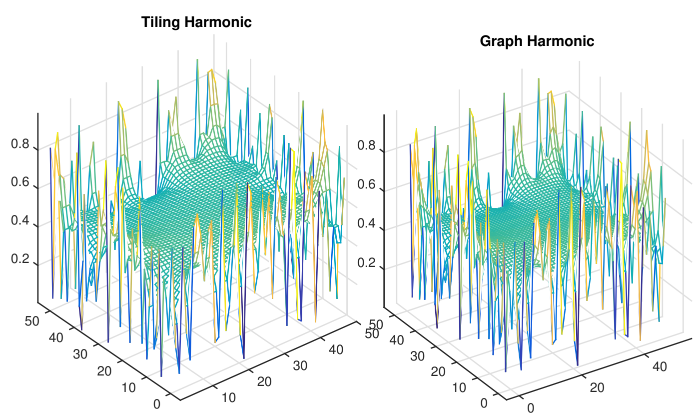
As many properties of graph-harmonic functions are known, it may be useful to compare them to tiling-harmonic functions, especially as the two functions are virtually identical on many sets of boundary values.

Figure 20: Half 0, Half 1 Boundary



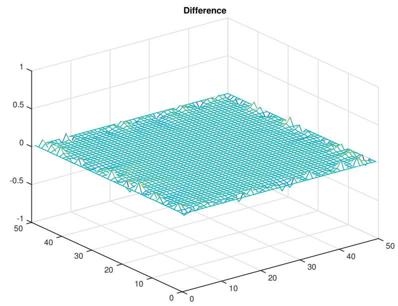
In Figure 20, which has boundary values half 0 and half 1, the two functions are almost indistinguishable.

Figure 21: Random Boundary



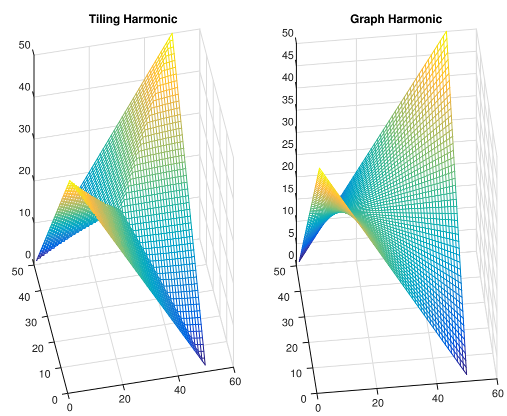
In Figure 21 as well, the two functions are nearly the same, only differing by a small amount near the boundary, as shown in Figure 22. For these figures, the boundary values are chosen uniformly at random from the interval $[0, 1]$. Notice both functions in Figure 21 are very close to flat in the center. This is because the random boundary values tend to oscillate — see Theorem 7. Thus for "most" sets of boundary values, tiling and graph-harmonic functions are virtually identical.

Figure 22: Random Boundary Difference



In Figure 23, however, the two functions have a nontrivial difference. The graph-harmonic function is smooth, while the tiling-harmonic one is made up of four flat triangles. This nontrivial difference was discovered in [1].

Figure 23: A Non-Trivial Difference



A proof of the near identicality of the two functions except in special cases as above would likely allow us to easily transfer many of the properties of graph-harmonic functions to tiling-harmonic ones.

6 Conjectures

The following two conjectures were given by Merenkov. Working toward proofs of these conjectures are the major directions of further work on tiling-harmonic functions.

Conjecture 1 (Liouville's Theorem for Tiling-Harmonic Functions)**.** *A bounded tiling-harmonic function on the regular lattice grid* ($\mathbb{Z}^2$) *must be constant.*

Liouville's Theorem is a major property of harmonic functions, and it would be a useful attribute to prove for tiling-harmonic functions. This conjecture also serves as a "simpler version" of the second, more important one.

Conjecture 2. *A nonnegative tiling-harmonic function on the upper half-plane that vanishes along the x-axis must be proportional to y.*

This conjecture is the motivation for studying tiling-harmonic functions; its proof would simplify many of the more involved arguments in [2].

7 Future Work

It would be useful to improve the bound in our proof of Harnack's Inequality for Tiling-Harmonic Functions, as a strong enough bound would lead to a proof of Conjecture 1, Liouville's Theorem.

Another approach toward a proof of Conjecture 1 is to extend Theorem 12 from a set with one accumulation point to a set with finitely many accumulation points. Conjecture 1 corresponds to a set with a countable number of accumulation points, as there are only a countable number of points on the integer lattice.

Additionally, it would be helpful to prove a connection between graph and tiling-harmonic functions, i.e., that they differ by a bounded amount except in certain special cases like in Figure 23, as it could then be possible to use this connection to prove tiling-harmonic versions of results known for graph-harmonic functions.

In regards to the more important Conjecture 2, there is an analogue of Harnack's Inequality suited toward this problem, with a boundary on which the function vanishes. It is called the Boundary Harnack Principle, and for any efforts on this conjecture, an exploration of this would be the first step. It has been proven for other classes of functions, but the Boundary Harnack Principle for Tiling-Harmonic Functions remains open.

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