# A Connection Between Vector Bundles over Smooth Projective Curves and Representations of Quivers

Dhruv W. Rohatgi
The Davidson Academy of Nevada

October 31, 2016

**Abstract**

We create a partition bijection that yields a partial result on a recent conjecture by Schiffmann relating the problems of counting over a finite field (1) vector bundles over smooth projective curves, and (2) representations of quivers.

## 1 Introduction

A vector bundle on a curve is a family of vector spaces, one associated with each element of the curve, that is "continuous" in a certain way. A representation of a quiver - that is, a directed graph - assigns one vector space to each vertex, and one linear transformation from the source vector space to the destination vector space for each edge. The analogy between the former and the latter quantity is one motivation for Schiffmann's work [4] in counting isomorphism classes of geometrically indecomposable vector bundles on a curve over finite fields, since in previous works Kac [2], Hua [1], and others have counted isomorphism classes of absolutely indecomposable representations of quivers over finite fields.

Let $\Sigma_g$ be the quiver with one vertex and $g$ edges. Schiffmann conjectures [4] that the quantity $A_{g,r,d}(0)$, arising from the problem of counting isomorphism classes of geometrically indecomposable vector bundles of rank $r$ and degree $d$ over a smooth projective curve of genus $g$, is equal to the value at $q = 1$ of $A_{\Sigma_g,r}$, Kac's A-polynomial in the variable $q$ counting the number of absolutely indecomposable representations of $\Sigma_g$ over $\mathbb{F}_q$ of dimension $r$. As further motivation, Schiffmann notes that the only difference between the formulas for $A_{g,r,d}(0)$ and $A_{\Sigma_g,r}(1)$ is a term $-l(\lambda)$ in an intermediate formula of the former quantity.

## 2 Basic definitions

We will use the definitions in a paper by Hua [1] for $A_{\Sigma_g}(\alpha, q)$ and $H_{\Sigma_g}(\alpha, q)$ in $\mathbb{Q}[q]$, and power series $P_{\Sigma_g}(X_1, \ldots, X_n, q)$ in $X_1, \ldots, X_n$ with coefficients in $\mathbb{Q}[q]$. Since $\Sigma_g$ has only one vertex, the dimension vector $\alpha$ is simply a positive integer and $n = 1$. For convenience we will not use $n$ again in this context, and will refer to $P_{\Sigma_g}(X_1, q)$ simply as $P_{\Sigma_g}(X, q)$. Note that $A_{\Sigma_g}(k, q) = A_{\Sigma_g, r}(q)$.

We now state the formula for $A_{g,r,d}(0)$ in a manner which mirrors Hua's formula for $A_{\Sigma_g}(k, q)$.

**Definition.** Following Hua [1] and Macdonald [3], let $\phi_l(q) = (1-q)(1-q^2)\cdots(1-q^l)$ for any $l \geq 0$, and let $b_\lambda(q) = \prod_{i \geq 1} \phi_{m^i}(q)$ for any $\lambda = (1^{m^1} 2^{m^2} \cdots) \in \mathscr{P}$.

**Definition.** Let $P'(X, q)$ be the generating function in $X$ with coefficients in $\mathbb{Q}(q)$ such that

$$P'_{\Sigma_g}(X, q) = \sum_{\pi \in \mathscr{P}} \frac{q^{(g-1)\langle \pi, \pi \rangle - l(\pi)}}{b_\pi(q^{-1})} X^{|\pi|}.$$

*Remark.* The only difference with $P_{\Sigma_g}(X, q)$ is the $-l(\pi)$ term, since we have

$$P_{\Sigma_g}(X, q) = \sum_{\pi \in \mathscr{P}} \frac{q^{(g-1)\langle \pi, \pi \rangle}}{b_\pi(q^{-1})} X^{|\pi|}.$$

**Definition.** For any $n > 0$, let $H'_{\Sigma_g}(n, q)$ be the rational function in $q$ determined by

$$\log(P'_{\Sigma_g}(X, q)) = \sum_{n > 0} H'_{\Sigma_g}(n, q) X^n / n.$$

**Definition.** For any $n > 0$, define

$$A'_{\Sigma_g}(n, q) = \frac{q-1}{n} \sum_{d|n} \mu(d) H'_{\Sigma_g}(\frac{n}{d}, q^d).$$

*Remark.* $H_{\Sigma_g}(n, q)$ and $A_{\Sigma_g}(n, q)$ are defined analogously to $H'_{\Sigma_g}(n, q)$ and $A'_{\Sigma_g}(n, q)$ but in terms of $P(X, q)$ rather than $P'(X, q)$.

Schiffmann conjectures that $A^0_{g,r}(z)$ is regular outside $z = 1$; in this case we have

$$A_{g,n,d}(0) = A'_{\Sigma_g}(n, 1).$$

Motivated by this conjecture, from here on we will ignore $A_{g,n,d}(0)$, and attempt to show that $A'_{\Sigma_g}(n, 1) = A_{\Sigma_g}(n, 1)$.

We also define several quantities involving both papers.

**Definition.** Let $f_{n,g}(q) = q^n A'_{\Sigma_g}(n, q) - A_{\Sigma_g}(n, q)$.

**Definition.** Let $h_{n,g}(q) = q^n H'_{\Sigma_g}(n, q) - H_{\Sigma_g}(n, q)$.

**Definition.** Let $B_k$ be the rational function in $q$ such that

$$\sum_{k=0}^{\infty} B_k X^k = \frac{P'_{\Sigma_g}(qX, q)}{P_{\Sigma_g}(X, q)}.$$

## 3 Schiffmann's Little Conjecture

**Conjecture 1.** *Schiffmann conjectures that* $A_{\Sigma_g}(n,1) = A'_{\Sigma_g}(n,1)$ *for all* $n$ *and* $g$*.*

*Remark.* Conjecture 5 is equivalent to the statement $q-1 \mid f_{n,g}$ for all $n$ and $g$.

**Proposition 2.** *If* $B_k$ *is a polynomial for all* $k$*, then* $f_{n,g}$ *is a multiple of* $q-1$ *for all* $n$*.*

*Proof.* Suppose that $B_k$ is a polynomial for all $k$. This means that the coefficients of

$$\frac{P'_{\Sigma_g}(qX,q)}{P_{\Sigma_g}(X,q)}$$

are polynomials. Applying logarithm to this power series preserves the property, since coefficients are only multiplied, scaled, and added. Hence,

$$\begin{aligned}
\sum_{k=0}^{\infty} h_{k,g}(q)X^k &= \sum_{k=0}^{\infty}(q^k H'_{\Sigma_g}(k,q) - H_{\Sigma_g}(k,q))X^k \\
&= \sum_{k=0}^{\infty} q^k H'_{\Sigma_g}(k,q)X^k - \sum_{k=0}^{\infty} H_{\Sigma_g}(k,q)X^k \\
&= \sum_{k=0}^{\infty} H'_{\Sigma_g}(k,q)(qX)^k - \sum_{k=0}^{\infty} H_{\Sigma_g}(k,q)X^k \\
&= \log P'_{\Sigma_g}(qX,q) - \log P_{\Sigma_g}(X,q) \\
&= \log \frac{P'_{\Sigma_g}(qX,q)}{P_{\Sigma_g}(X,q)}
\end{aligned}$$

is also a power series of polynomials. In other words, $h_{k,g}(q)$ is a polynomial for all $k$.
But then

$$\begin{aligned}
f_{n,g}(q) &= q^n A'_{\Sigma_g}(n,q) - A_{\Sigma_g}(n,q) \\
&= q^n \frac{q-1}{n} \sum_{d|n} \left(\mu(d) H'_{\Sigma_g}(\frac{n}{d}, q^d)\right) - \frac{q-1}{n} \sum_{d|n} \left(\mu(d) H_{\Sigma_g}(\frac{n}{d}, q^d)\right) \\
&= \frac{q-1}{n} \sum_{d|n} \left(q^n \mu(d) H'_{\Sigma_g}\left(\frac{n}{d}, q^d\right) - \mu(d) H_{\Sigma_g}\left(\frac{n}{d}, q^d\right)\right) \\
&= \frac{q-1}{n} \sum_{d|n} \mu(d) \left(q^n H'_{\Sigma_g}\left(\frac{n}{d}, q^d\right) - H_{\Sigma_g}\left(\frac{n}{d}, q^d\right)\right) \\
&= \frac{q-1}{n} \sum_{d|n} \mu(d) \left(\left(q^{\frac{n}{d}} H'_{\Sigma_g}\right)\left(\frac{n}{d}, q^d\right) - H_{\Sigma_g}\left(\frac{n}{d}, q^d\right)\right) \\
&= \frac{q-1}{n} \sum_{d|n} \mu(d) \left(q^{\frac{n}{d}} H'_{\Sigma_g} - H_{\Sigma_g}\right)\left(\frac{n}{d}, q^d\right)
\end{aligned}$$

$$= \frac{q-1}{n} \sum_{d|n} \mu(d) h_{\frac{n}{d},g}(q^d)$$

is a multiple of $q-1$. $\square$

**Conjecture 3.** *$B_k$ is a polynomial for all $k$.*

*Remark.* Proposition 2 shows that our Conjecture 3 would imply Conjecture 5. Computational evidence suggested that $h_{k,g}(q)$ is a polynomial for all $k$, which is the equivalent statement to Conjecture 3 that we used in the proof of Proposition 2.

**Definition.** We define for any partition $\pi$,

$$C_\pi = \frac{q^{(g-1)\langle \pi,\pi\rangle}}{b_\pi(q^{-1})}.$$

We now obtain a more useful formula for $B_k$.

**Proposition 4.** *For any $k$,*

$$B_k = \sum_{\pi_0,\dots,\pi_s} q^{|\pi_0|-l(\pi_0)}(-1)^s \prod_{i=0}^{s} C_{\pi_i}$$

*where $\pi_0$ may be the empty partition but $\pi_1,\dots,\pi_s$ are all nonempty, and $\sum_{i=0}^{s} |\pi_i| = k$.*

*Proof.* We let

$$C_k = \sum_{\pi \,\middle|\, |\pi|=k} C_\pi$$

and

$$C'_k = \sum_{\pi \,\middle|\, |\pi|=k} q^{k-l(\pi)} C_\pi.$$

Then we have

$$\begin{aligned}
\sum_{k=0}^{\infty} B_k X^k &= \frac{P'_{\Sigma_g}(qX,q)}{P_{\Sigma_g}(X,q)} \\
&= \frac{\sum_{k=0}^{\infty} C'(k)X^k}{\sum_{k=0}^{\infty} C(k)X^k} \\
&= \frac{\sum_{k=0}^{\infty} C'(k)X^k}{1+\sum_{k=1}^{\infty} C(k)X^k} \\
&= \left(\sum_{k=0}^{\infty} C'(k)X^k\right)\left(1-\left(\sum_{k=1}^{\infty} C(k)X^k\right)+\left(\sum_{k=1}^{\infty} C(k)X^k\right)^2-\dots\right)
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \left( C'(k-n) \sum_{\substack{l_1+\cdots+l_s=n \\ l_1,\ldots,l_s>0}} \prod_{i=1}^{s} -C(l_i) \right) X^k$$

$$= \sum_{k=0}^{\infty} \left( \sum_{\substack{l_0+l_1+\cdots+l_s=k \\ l_1,\ldots,l_s>0}} C'(l_0) \prod_{i=1}^{s} -C(l_i) \right) X^k$$

$$= \sum_{k=0}^{\infty} \left( \sum_{\substack{\pi_0,\ldots,\pi_s \in \mathscr{P} \\ |\pi_0|+\cdots+|\pi_s|=k \\ |\pi_1|,\ldots,|\pi_s|>0}} C'_{\pi_0} \prod_{i=1}^{s} -C_{\pi_i} \right) X^k$$

This yields the desired result. $\square$

**Definition.** Based on the above formula, we define for any partition $\pi$,

$$B_\pi = \sum_{\substack{\pi_0 \cup \cdots \cup \pi_s = \pi \\ |\pi_1|,\ldots,|\pi_s|>0}} q^{|\pi_0|-l(\pi_0)} (-1)^s \prod_{i=0}^{s} C_{\pi_i}.$$

**Conjecture 5.** *$B_\pi$ is in fact a polynomial for all partitions $\pi$.*

*Remark.* The above conjecture implies Conjecture 3, since $B_k$ is the sum of $B_\pi$ over partitions $\pi$ of size $k$. Once again, we made this (unexpected) conjecture on the basis of computer evidence.

# 4 Combinatorial Interpretation of $B_\pi$

## 4.1 Flat Partition Case

We are now ready to state and prove the main result which we have achieved thus far.

**Theorem 6.** *Let $g \geq 1$ and $\pi$ be the partition with $b$ elements, each of size $a$. Then $B_\pi$ is a polynomial.*

*Proof.* We have

$$B_\pi = \sum_{\substack{l_0+\cdots+l_s=b \\ l_1,\ldots,l_s>0}} q^{al_0-l_0} (-1)^s \prod_{i=0}^{s} \frac{q^{(g-1)al_i^2}}{(1-q^{-1})\cdots(1-q^{-l_i})}$$

$$= \sum_{\substack{l_0+\cdots+l_s=b \\ l_1,\ldots,l_s>0}} q^{al_0-l_0} (-1)^s \prod_{i=0}^{s} \frac{q^{(g-1)al_i^2} q^{\frac{l_i(l_i+1)}{2}}}{(q-1)\cdots(q^{l_i}-1)}$$

$$= q^{-(g-1)ab} \sum_{\substack{l_0+\cdots+l_s=b \\ l_1,\ldots,l_s>0}} q^{al_0-l_0}(-1)^s \prod_{i=0}^{s} \frac{q^{(g-1)al_i(l_i+1)} q^{\frac{l_i(l_i+1)}{2}}}{(q-1)\cdots(q^{l_i}-1)}$$

$$= q^{-(g-1)ab} \sum_{\substack{l_0+\cdots+l_s=b \\ l_1,\ldots,l_s>0}} q^{al_0-l_0}(-1)^s \prod_{i=0}^{s} \frac{q^{(2(g-1)a+1)\frac{l_i(l_i+1)}{2}}}{(q-1)\cdots(q^{l_i}-1)}$$

$$= (-1)^b q^{-(g-1)ab} \sum_{\substack{l_0+\cdots+l_s=b \\ l_1,\ldots,l_s>0}} q^{al_0-l_0}(-1)^s \prod_{i=0}^{s} \frac{q^{(2(g-1)a+1)\frac{l_i(l_i+1)}{2}}}{(1-q)\cdots(1-q^{l_i})}$$

Guided by the above equation, we let $d = 2(g-1)a+1$. We know that

$$\frac{1}{(1-q)\cdots(1-q^l)}$$

denotes the number of partitions (possible empty) of length at most $l$, where the weight of a partition $\lambda$ is $q^{|\lambda|}$. Hence,

$$\frac{q^{d\frac{l(l+1)}{2}}}{(1-q)\cdots(1-q^l)}$$

denotes the number of partitions $\lambda$ of length exactly $l$ such that $\lambda_i - \lambda_{i+1} \geq d$ for each consecutive pair, and $\lambda_l \geq d$. This is because the term $q^{d\frac{l(l+1)}{2}}$ essentially adds $d(l+1-i)$ to the $i^{\text{th}}$ term of $\lambda$. Again, the weight of a partition $\lambda$ is $q^{|\lambda|}$. We call $\lambda$ a *$d$-stair* partition.

It follows that $(-1)^b q^{(g-1)ab} B_\pi$ is a generating function in $q$ that counts the number of tuples of $d$-stair partitions where only the first partition may be empty, and the total length of the tuple is $b$. The weight of a tuple $p = (p^0, p^1, p^2, \ldots, p^s)$ is

$$w(p) = (-1)^s q^{(a-1)l(p^0)+|p^0|+|p^1|+\cdots+|p^s|}.$$

To show that $B_\pi$ is a polynomial, it suffices to show that $(-1)^b q^{(g-1)ab} B_\pi$ is a polynomial, since $B_\pi$ has no poles at $q=0$. Let $S_0 = b(b+1)(g-1)a + b(a-1)$. Then we must show that for all $S > S_0$, the tuples with absolute weight $q^S$ cancel out. We fix $S > S_0$. From here on we will only consider tuples of $d$-stair partitions with total length $b$ and weight $S$.

First, we cancel out partition tuples where the first partition is nonempty. Consider any such tuple $(p^0, p^1, p^2, \ldots, p^s)$. The corresponding tuple is $((\,), p^0 + a-1, p^1, p^2, \ldots, p^s)$, where $(\,)$ is the empty partition, and $p^0+a-1$ is the partition given by adding $a-1$ to each element of $p^0$. It is clear that this is a bijection between tuples with first partition nonempty, and tuples such that the first partition is empty and the second partition's last element is at least $d+a-1$. Furthermore, since we decreased the length of $p^0$ by $l(p^0)$ but increased the total size by $(a-1)l(p^0)$, the weight of the original tuple is the same by absolute value as the corresponding tuple. It also has opposite sign: we increased the number of partitions by 1 (though the total length remains $b$). Thus these tuples cancel out. We will refer later to this bijection as $f_0$.

Now we are left with the tuples such that the first partition is empty, and the second partition's last element is less than $d+a-1$. For convenience we drop the initial empty partition, and consider all tuples of nonempty partitions where the first partition's last element is less than $d+a-1$. The absolute weight is the total size of the partitions, and the sign is $(-1)^{s+1}$, where $s$ is the number of partitions.

Before defining a bijection that cancels these out, we define two basic functions on such partition tuples: *unroll* and *tuck*.

We *unroll* one partition $(c_1, \ldots, c_n, c_{n+1})$ of a partition tuple by replacing this partition by two consecutive partitions $(c_{n+1})$ and $(c_1, \ldots, c_n)$. This operation trivially preserves the $d$-stair property. Note that the total length of the tuple is still $b$, and the last element of the first partition is unchanged.

For example, given the partition tuple

$$((5), (30, 10, 5))$$

we can unroll the second partition and obtain

$$((5), (5), (30, 10)).$$

We *tuck* one singleton partition $(c)$ into the next partition $(c_1, \ldots, c_n)$ by replacing these two $d$-stair partitions by one $d$-stair partition $(c_1, \ldots, c_n, c)$. Thus, we require that $c_n - c \geq d$. Once again, note that the total length of the tuple is still $b$, and the last element of the first partition is unchanged.

For example, given the partition tuple

$$((6), (10), (20, 15), (25))$$

with $d = 5$ we may not tuck the first partition or third partition, but we may tuck the second partition to obtain

$$((6), (20, 15, 10), (25)).$$

We are now ready to define the involution $f$ on the set of tuples of nonempty partitions with the last element of the first partition less than $d+a-1$. Consider any tuple $p = (p^1, p^2, \ldots, p^s)$. Let $i$ be the index of the first partition which can be tucked, and let $j$ be the index of the first partition which can be unrolled. If no such $i$ or no such $j$ exists, we assign $i = s+1$ or $j = s+1$.

Either $i \leq s$ or $j \leq s$. Suppose not; then every partition is a singleton, and the difference between consecutive elements is at most $d-1$. Hence,

$$\begin{aligned} S &= |p^1| + |p^2| + \cdots + |p^b| \\ &\leq (a+d-2) + (a+d-2+(d-1)) + \cdots + (a+d-2+(b-1)(d-1)) \end{aligned}$$

$$\begin{aligned}
&= b(a+d-2) + \frac{b(b-1)}{2}(d-1) \\
&= b(a+2(g-1)a-1) + b(b-1)(g-1)a \\
&= b(a-1) + b(b+1)(g-1)a. \\
&= S_0
\end{aligned}$$

Contradiction, since we assumed that $S > S_0$.

Furthermore, note that $i \neq j$, since only singleton partitions may be tucked and only multiple partitions may be unrolled.

If $i < j$, we define $f(p)$ to be the tuple $p$ with $p^i$ tucked into $p^{i+1}$. That is,

$$f(p) = (p^1, p^2, \ldots, p^{i-1}, p^{i+1} \cup p^i, p^{i+2}, \ldots, p^s).$$

If $i > j$ then we define $f(p)$ to be the tuple $p$ with $p^j$ unrolled. So if $p^j = (c_1, \ldots, c_n)$ then

$$f(p) = (p^1, p^2, \ldots, p^{j-1}, (c_n), (c_1, \ldots, c_{n-1}), p^{j+1}, \ldots, p^s).$$

Suppose $i < j$. Then $f(p)^i$ can be unrolled. Every earlier partition in $p$ was a singleton, so no earlier partition in $f(p)$ can be unrolled. Furthermore, none of $f(p)^1, \ldots, f(p)^{i-2}$ can be tucked, since the first $i-1$ elements of $f(p)$ are equal to the first $i-1$ elements of $p$. Finally, $f(p)^{i-1} = p^{i-1}$ cannot be tucked into $f(p)^i = p^{i+1} \cup p^i$, since otherwise $p^{i-1}$ could have been tucked into $p^i$. Hence, in applying $f$ to $f(p)$ the partition $f(p)^i$ will be unrolled, so $f(f(p)) = p$.

Suppose $i > j$. Then $f(p)^j$ can be tucked. Every earlier partition in $p$ was a singleton, so no earlier partition in $f(p)$ can be unrolled. Furthermore, none of $f(p)^1, \ldots, f(p)^{j-2}$ can be tucked, since the first $j-1$ elements of $f(p)$ are equal to the first $j-1$ elements of $p$. Finally, $f(p)^{j-1} = p^{j-1}$ cannot be tucked into $f(p)^j$, the partition consisting of the last element of $p^j$, since otherwise $p^{j-1}$ could have been tucked into $p^j$. Hence, in applying $f$ to $f(p)$ the partition $f(p)^j$ will be tucked, so once again $f(f(p)) = p$.

We conclude that $f$ is an involution, so all tuples which were not cancelled by $f_0$ can be grouped in pairs $\{p, f(p)\}$. Furthermore, the weights in each pair have opposite sign, since $f$ either increases or decreases the number of partitions in $p$ by 1. It follows that all tuples of absolute weight $q^S$ cancel.

Thus, $(-1)^b q^{(g-1)ab} B_\pi$ represents a sequence with no nonzero elements beyond $S_0$, so $(-1)^b q^{(g-1)ab} B_\pi$ is a polynomial and therefore $B_\pi$ is a polynomial. □

**Corollary.** *Let $\pi$ be the partition with $b$ elements, each of size $a$. Then the degree of $B_\pi$ is $b^2(g-1)a + b(a-1)$.*

*Proof.* It immediately follows from the proof of Theorem 6 that

$$\deg(-1)^b q^{(g-1)ab} B_\pi \leq b(b+1)(g-1)a + b(a-1)$$

so

$$\deg B_\pi \leq b^2(g-1)a + b(a-1).$$

The reverse inequality follows since there is exactly one tuple with size $n = b(b+1)(g-1)a + b(a-1)$ that cannot be unrolled nor tucked, so there is an odd number of tuples of size $n$: thus the coefficient of $q^n$ in $(-1)^b q^{(g-1)ab} B_\pi$ is necessarily nonzero. $\square$

## 4.2 General Case

We now generalize the combinatorial interpretation of $B_\pi$ to all partitions.

**Definition.** Let $\lambda = (\lambda^1, \lambda^2, \lambda^3, \dots)$ be a sequence of partition tuples, where each tuple has the same number of (possibly empty) partitions, some $s+1$. Set $m_i^a = l(\lambda_i^a)$ for each $i$ and $a$, and set

$$t_i^a = a + (a-1)m_i^{a-1} + (a-2)m_i^{a-2} + \cdots + m_i^1.$$

Suppose that the following conditions also hold:

1. $\sum_{a=1}^{\infty} m_i^a > 0$ for each $i > 0$
2. in partition $\lambda_i^a$ the difference between consecutive elements is at least $2(g-1)a+1$
3. in partition $\lambda_i^a$ the smallest element is at least $2(g-1)t_i^a + 1$

Then we call $\lambda$ a *stair* sequence.

**Theorem 7.** *Let $\pi = (1^{m^1} 2^{m^2} \dots)$ be any partition. Then*

$$\frac{B_\pi}{(-1)^{l(\pi)} q^{-(g-1)|\pi|}}$$

*is the sum over all stair sequences $\lambda$ with $\sum_{i=0}^{s} m_i^a = m^a$ for each $a > 0$, of*

$$w(\lambda) = (-1)^s \prod_{a=1}^{\infty} q^{(a-1)m_0^a + |\lambda_0^a| + \cdots + |\lambda_s^a|}.$$

*Proof.* Let $\pi_0, \pi_1, \dots, \pi_s$ be any partitions with $\pi_1, \dots, \pi_s$ nonempty and $\pi_0 \cup \pi_1 \cup \cdots \cup \pi_s = \pi$.

For each $\pi_i$ we let the exponential form be

$$\pi_i = (1^{m_i^1} 2^{m_i^2} 3^{m_i^3} \dots).$$

And we let the exponential form of $\pi$ be

$$\pi = (1^{m^1} 2^{m^2} 3^{m^3} \dots).$$

Note that

$$\langle \pi_i, \pi_i \rangle = \sum_{j=1}^{l(\pi_i')} (\pi_{i_j}')^2$$

$$
\begin{aligned}
&= -|\pi_i| + \sum_{j=1}^{l(\pi_i')} \pi_{i_j}'(\pi_{i_j}' + 1) \\
&= -|\pi_i| + 2\sum_{j=1}^{l(\pi_i')} (\pi_{i_j}' + \cdots + 1) \\
&= -|\pi_i| + 2\sum_{j=1}^{l(\pi_i)} \sum_{k=j}^{l(\pi_i)} \pi_{i_k} \\
&= -|\pi_i| + 2\sum_{a=1}^{\infty} \sum_{j=1}^{m_i^a} (aj + (a-1)m_i^{a-1} + (a-2)m_i^{a-2} + \cdots + m_i^1) \\
&= -|\pi_i| + 2\sum_{a=1}^{\infty} \sum_{j=1}^{m_i^a} a(j-1) + t_i^a \\
&= -|\pi_i| + 2\sum_{a=1}^{\infty} \sum_{j=0}^{m_i^a-1} aj + t_i^a
\end{aligned}
$$

where

$$
t_i^a = a + (a-1)m_i^{a-1} + (a-2)m_i^{a-2} + \cdots + m_i^1.
$$

Then the term in $B_\pi$ corresponding to $(\pi_0, \ldots, \pi_s)$ is

$$
\begin{aligned}
B_{(\pi_0,\ldots,\pi_s)} &= q^{|\pi_0| - l(\pi_0)} (-1)^s \prod_{i=0}^{s} C_{\pi_i} \\
&= q^{\sum_{a=1}^{\infty} m_0^a (a-1)} (-1)^s \prod_{i=0}^{s} \frac{q^{(g-1)\langle \pi_i, \pi_i \rangle}}{b_\pi(q^{-1})} \\
&= q^{\sum_{a=1}^{\infty} m_0^a (a-1)} (-1)^s \prod_{i=0}^{s} \frac{q^{-(g-1)|\pi_i| + 2(g-1)\sum_{a=1}^{\infty} \sum_{j=0}^{m_i^a-1} a + t_i^a}}{\prod_{a=1}^{\infty} (1-q^{-1}) \cdots (1-q^{-m_i^a})} \\
&= q^{-(g-1)|\pi|} q^{\sum_{a=1}^{\infty} m_0^a (a-1)} (-1)^s \prod_{i=0}^{s} \frac{q^{\sum_{a=1}^{\infty} 2(g-1) \sum_{j=0}^{m_i^a-1} a + t_i^a}}{\prod_{a=1}^{\infty} (1-q^{-1}) \cdots (1-q^{-m_i^a})} \\
&= q^{-(g-1)|\pi|} q^{\sum_{a=1}^{\infty} m_0^a (a-1)} (-1)^s \prod_{i=0}^{s} \prod_{a=1}^{\infty} \frac{q^{2(g-1)\sum_{j=0}^{m_i^a-1} aj + t_i^a}}{(1-q^{-1}) \cdots (1-q^{-m_i^a})} \\
&= q^{-(g-1)|\pi|} q^{\sum_{a=1}^{\infty} m_0^a (a-1)} (-1)^s \prod_{i=0}^{s} (-1)^{l(\pi_i)} \prod_{a=1}^{\infty} \frac{q^{2(g-1)\sum_{j=0}^{m_i^a-1} aj + t_i^a}}{(q^{-1}-1) \cdots (q^{-m_i^a}-1)} \\
&= q^{-(g-1)|\pi|} q^{\sum_{a=1}^{\infty} m_0^a (a-1)} (-1)^s \prod_{i=0}^{s} (-1)^{l(\pi_i)} \prod_{a=1}^{\infty} \frac{q^{\sum_{j=0}^{m_i^a-1} 2(g-1)aj + 2(g-1)t_i^a}}{(q^{-1}-1) \cdots (q^{-m_i^a}-1)}
\end{aligned}
$$

$$= q^{-(g-1)|\pi|} q^{\sum_{a=1}^{\infty} m_0^a(a-1)} (-1)^s \prod_{i=0}^{s} (-1)^{l(\pi_i)} \prod_{a=1}^{\infty} \frac{q^{\sum_{j=0}^{m_i^a-1}(2(g-1)a+1)j+2(g-1)t_i^a+1}}{(1-q)\cdots(1-q^{m_i^a})}$$

$$= (-1)^{l(\pi)} q^{-(g-1)|\pi|} q^{\sum_{a=1}^{\infty} m_0^a(a-1)} (-1)^s \prod_{i=0}^{s} \prod_{a=1}^{\infty} \frac{q^{\sum_{j=0}^{m_i^a-1}(2(g-1)a+1)j+2(g-1)t_i^a+1}}{(1-q)\cdots(1-q^{m_i^a})}$$

$$= (-1)^{l(\pi)} q^{-(g-1)|\pi|} (-1)^s \prod_{a=1}^{\infty} q^{m_0^a(a-1)} \prod_{i=0}^{s} \frac{q^{\sum_{j=0}^{m_i^a-1}(2(g-1)a+1)j+2(g-1)t_i^a+1}}{(1-q)\cdots(1-q^{m_i^a})}$$

The term inside the first product counts, for a fixed $a$, the partition tuples $\lambda^a = (\lambda_0^a, \ldots, \lambda_s^a)$ where $l(\lambda_i^a) = m_i^a$, and the difference between consecutive elements of a partition is at least $2(g-1)a+1$, and the smallest element of a partition is at least $2(g-1)t_i^a + 1$. The weight of $\lambda^a$ is

$$w(\lambda^a) = q^{(a-1)m_0^a + |\lambda_0^a| + \cdots + |\lambda_s^a|}.$$

Thus, we see that

$$\frac{B_{(\pi_0,\ldots,\pi_s)}}{(-1)^{l(\pi)} q^{-(g-1)|\pi|}}$$

counts sequences of partition tuples $\lambda = (\lambda^1, \lambda^2, \lambda^3, \ldots)$ such that each tuple satisfies the above conditions, with weight

$$w(\lambda) = (-1)^s \prod_{a=1}^{\infty} w(\lambda^a).$$

The theorem follows since $B_\pi$ is the sum of $B_{(\pi_0,\ldots,\pi_s)}$ over all possible values of the array $((m_i^a))$ with $\sum_{a=0}^{\infty} m_i^a > 0$ for each $i > 0$ - that is, $\pi_1, \ldots, \pi_s$ are nonempty - and $m^a = \sum_{i=0}^{s} m_i^a$ for each $a > 0$ - that is, $\pi_0 \cup \cdots \cup \pi_s = \pi$. $\square$

# 5 Direct Interpretation of $B_\pi$

**Definition.** Let the *decomposition set* of a partition $\pi$, denoted as $S(\pi)$, be the set of partition tuples $(\pi_0, \pi_1, \ldots, \pi_s)$ for any $s \geq 0$ where $\pi_0 \cup \cdots \cup \pi_s = \pi$ and $|\pi_i| > 0$ for $0 < i \leq s$.

Similarly, let the *positive decomposition set* of a partition $\pi$, denoted as $S'(\pi)$, be the set of partition tuples $(\pi_0, \pi_1, \ldots, \pi_s)$ for any $s \geq 0$ where $\pi_0 \cup \cdots \cup \pi_s = \pi$ and $|\pi_i| > 0$ for $0 \leq i \leq s$.

**Proposition 8.** *Let $\pi$ be a partition with $n$ distinct parts. Then*

$$\frac{(q-1)^n}{q^n} B_\pi = \sum_{(\pi_0,\ldots,\pi_s)\in S(\pi)} (-1)^s q^{|\pi_0| - l(\pi_0)} \prod_{i=0}^{s} q^{(g-1)\langle \pi_i, \pi_i \rangle}.$$

*Proof.* Note that for any $(\pi_0, \pi_1, \ldots, \pi_s) \in S(\pi)$ we have

$$\prod_{i=0}^{s} b_{\pi_i}(q^{-1}) = (1 - q^{-1})^n.$$

Therefore multiplying the definition of $B_\pi$ by this product yields the desired result. □

Hence $\frac{(q-1)^n}{q^n} B_\pi$ is a polynomial. To show that $B_\pi$ is a polynomial, we must show that $\frac{(q-1)^n}{q^n} B_\pi$ is a multiple of $(q-1)^n$. If $n = 1$ there is a trivial test for the divisibility of a polynomial $f(q)$—just check whether $f(1) = 0$. For $n > 1$, a more complex test is required.

**Lemma 9.** *Let*

$$f(q) = \sum_{i=0}^{\infty} a_i q^i$$

*be an arbitrary generating function in $q$. Then for each $n > 0$,*

$$\frac{f(q)}{(1-q)^n} = \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} a_j \binom{i-j+n-1}{n-1} \right) q^i.$$

*Proof.* For $n = 1$ we have:

$$\begin{aligned}
\frac{f(q)}{1-q} &= f(q)(1 + q + q^2 + \ldots) \\
&= \left( \sum_{i=0}^{\infty} a_i q^i \right) \left( \sum_{j=0}^{\infty} q^j \right) \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{i} a_j q^j q^{i-j} \\
&= \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} a_j \right) q^i.
\end{aligned}$$

Now suppose that the equation holds for some $n > 0$. Then:

$$\begin{aligned}
\frac{f(q)}{(1-q)^{n+1}} &= \left( \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} a_j \binom{i-j+n-1}{n-1} \right) q^i \right) \left( \sum_{i=0}^{\infty} q^i \right) \\
&= \sum_{i=0}^{\infty} \left( \sum_{k=0}^{i} \sum_{j=0}^{k} a_j \binom{k-j+n-1}{n-1} \right) q^i \\
&= \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} a_j \sum_{k=j}^{i} \binom{k-j+n-1}{n-1} \right) q^i
\end{aligned}$$

$$= \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} a_j \sum_{k=n-1}^{i-j+n-1} \binom{k}{n-1} \right) q^i$$

$$= \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} a_j \binom{i-j+n}{n} \right) q^i.$$

By induction, the equation holds for all $n > 0$. $\square$

**Lemma 10.** *Let*

$$f(q) = a_m q^m + \cdots + a_1 q + a_0$$

*be an arbitrary polynomial in $q$ and let $n \geq 0$. Then $(1-q)^n \mid f(q)$ if and only if*

$$\sum_{i=0}^{m} a_i \binom{i}{k} = 0$$

*for all $k$ with $0 \leq k < n$.*

*Proof.* ( $\Longrightarrow$ ) Suppose that $(1-q)^n \mid f(q)$. Then we also know that $(1-q)^n \mid q^m f(q^{-1})$ where

$$q^m f(q^{-1}) = a_0 q^m + \cdots + a_{m-1} q + a_m.$$

Let $k < n$; then

$$\frac{q^m f(q^{-1})}{(1-q)^{k+1}} = \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} a_{m-j} \binom{i-j+k}{k} \right) q^i$$

is a polynomial with degree $m-k-1$. Hence,

$$\sum_{j=0}^{m-k} a_{m-j} \binom{m-j}{k} = 0$$

and therefore

$$\sum_{i=k}^{m} a_i \binom{i}{k} = 0.$$

With the convention that $\binom{i}{k} = 0$ if $i < k$, it follows that

$$\sum_{i=0}^{m} a_i \binom{i}{k} = 0.$$

( $\Longleftarrow$ ) Now suppose that

$$\sum_{i=0}^{m} a_i \binom{i}{k} = 0$$

for all $k$ with $0 \leq k < n$. Define

$$g(p,k) = \sum_{j=0}^{m} a_j \binom{j+p-1}{k-1}$$

for $p \geq 1$ and $1 \leq k \leq n$. Then

$$g(1,k) = 0$$

and

$$g(p,1) = \sum_{j=0}^{m} a_j$$

by the assumption, and

$$g(p,k) = g(p-1,k) + g(p-1,k-1)$$

if $p,k > 1$. Therefore by induction, $g(p,k) = 0$ for all $p$ and $k$. In particular, we have for all $p \geq 1$:

$$\begin{aligned} 0 &= g(p,n) \\ &= \sum_{j=0}^{m} a_j \binom{j+p-1}{n-1} \\ &= \sum_{j=n-p}^{m} a_j \binom{j+p-1}{n-1} \\ &= \sum_{j=0}^{m-n+p} a_{m-j} \binom{m-j+p-1}{n-1}. \end{aligned}$$

Therefore

$$\frac{q^m f(q^{-1})}{(1-q)^n} = \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} a_{m-j} \binom{i-j+n-1}{n-1} \right) q^i$$

is a polynomial with degree at most $m-n$. It follows that $(1-q)^n \mid f(q)$. $\square$

Applying this lemma to $B_\pi$ yields the following equivalency.

**Proposition 11.** *Let $\pi$ be a partition with $n$ distinct parts. Then $B_\pi$ is a polynomial if and only if*

$$\sum_{(\pi_0,\ldots,\pi_s)\in S(\pi)} (-1)^s \binom{|\pi_0| - l(\pi_0) + (g-1)(\langle \pi_0, \pi_0\rangle + \cdots + \langle \pi_s, \pi_s\rangle)}{k} = 0$$

*for all $k$ with $0 \leq k < n$.*

*Proof.* Let
$$\frac{(q-1)^n}{q^n}B_\pi = a_m q^m + \cdots + a_1 q + a_0.$$
We have
$$\begin{aligned}\frac{(q-1)^n}{q^n}B_\pi &= \sum_{(\pi_0,\ldots,\pi_s)\in S(\pi)} (-1)^s q^{|\pi_0|-l(\pi_0)} \prod_{i=0}^{s} q^{(g-1)\langle\pi_i,\pi_i\rangle} \\ &= \sum_{(\pi_0,\ldots,\pi_s)\in S(\pi)} (-1)^s q^{|\pi_0|-l(\pi_0)+(g-1)\sum_{i=0}^{s}\langle\pi_i,\pi_i\rangle}.\end{aligned}$$

For any tuple $\alpha = (\pi_0,\ldots,\pi_s)$ let $\mathrm{coef}(\alpha) = (-1)^s$ and let $\exp(\alpha) = |\pi_0| - l(\pi_0) + (g-1)\sum_{i=0}^{s}\langle\pi_i,\pi_i\rangle$. Then

$$\begin{aligned}\sum_{i=0}^{m} a_i\binom{i}{k} &= \sum_{i=0}^{m}\left(\sum_{\alpha\in S(\pi)|\exp(\alpha)=i}\mathrm{coef}(\alpha)\right)\binom{i}{k} \\ &= \sum_{i=0}^{m}\left(\sum_{\alpha\in S(\pi)|\exp(\alpha)=i}\mathrm{coef}(\alpha)\binom{\exp(\alpha)}{k}\right) \\ &= \sum_{\alpha\in S(\pi)}\mathrm{coef}(\alpha)\binom{\exp(\alpha)}{k} \\ &= \sum_{(\pi_0,\ldots,\pi_s)\in S(\pi)}(-1)^s\binom{|\pi_0|-l(\pi_0)+(g-1)(\langle\pi_0,\pi_0\rangle+\cdots+\langle\pi_s,\pi_s\rangle)}{k}.\end{aligned}$$

Applying Lemma 10 completes the proof. □

If we consider any particular pair $(\pi, g)$, where $\pi$ is a partition of length $n$, to be a point in $\mathbb{R}_{n+1}$, then the left-hand side of the condition in Proposition 11 is a particular polynomial in $n+1$ variables, evaluated at $(\pi, g)$. Thus we can rephrase the infinite conjectured scalar equalities of Proposition 11 into one polynomial equality.

**Proposition 12.** *Let $n$ be a positive integer. Then $B_\pi$ is a polynomial for each partition with $n$ distinct elements and each $g > 0$, if and only if*
$$\sum_{(\pi_0,\ldots,\pi_s)\in S(\pi)}(-1)^s\binom{|\pi_0|-l(\pi_0)+(g-1)(\langle\pi_0,\pi_0\rangle+\cdots+\langle\pi_s,\pi_s\rangle)}{k} = 0$$
*where $0 \le k < n$, and $\pi$ and $g$ are variables.*

*Proof.* The sum can be interpreted as a polynomial provided that the combinatorial definition of the binomial coefficient matches the algebraic definition
$$\binom{n_1}{n_2} = \frac{n_1(n_1-1)\cdots(n_1-n_2+1)}{n_2(n_2-1)\cdots 1}$$

for all coefficients of the form

$$\binom{|\pi_0| - l(\pi_0) + (g-1)(\langle \pi_0, \pi_0\rangle + \cdots + \langle \pi_s, \pi_s\rangle)}{k}$$

where $\pi$ is a partition of $n$ distinct elements, $g$ is a positive integer, $k$ is an integer between 0 and $n-1$, and $(\pi_0, \ldots, \pi_s) \in S(\pi)$. But since $k$ and $|\pi_0| - l(\pi_0) + (g-1)(\langle \pi_0, \pi_0\rangle + \cdots + \langle \pi_s, \pi_s\rangle)$ are always nonnegative integers, the definitions do indeed match. $\square$

Now we convert the binomial coefficients in Proposition 11 into exponentials, which allow further simplifications, using the following lemma essentially stating that the binomial coefficients

$$\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{k}$$

span the space of polynomials with degree at most $k$.

**Lemma 13.** *For any integer $k \geq 0$,*

$$x^k \in \operatorname{span}\left\{\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{k}\right\}$$

*and*

$$\binom{x}{k} \in \operatorname{span}\{1, x, \ldots, x^k\}.$$

**Proposition 14.** *Let $n$ be a positive integer. Then $B_\pi$ is a polynomial for each partition with $n$ distinct elements and each $g > 0$, if and only if*

$$\sum_{(\pi_0,\ldots,\pi_s)\in S(\pi)} (-1)^s \left(|\pi_0| - l(\pi_0) + (g-1)(\langle \pi_0, \pi_0\rangle + \cdots + \langle \pi_s, \pi_s\rangle)\right)^k = 0$$

*for all $k$ with $0 \leq k < n$.*

**Proposition 15.** *Let $n$ be a positive integer. Then $B_\pi$ is a polynomial for each partition with $n$ distinct elements and each $g > 0$, if and only if*

$$\sum_{(\pi_0,\ldots,\pi_s)\in S'(\pi)} (-1)^s (|\pi_0| - l(\pi_0))^i (\langle \pi_0, \pi_0\rangle + \cdots + \langle \pi_s, \pi_s\rangle)^{k-i} = 0$$

*for all $i$ and $k$ with $0 < i \leq k < n$.*

*Proof.* The condition in Proposition 14 is equivalent to the following conditions together, which are the result of applying the binomial theorem and grouping terms with the same power of $g-1$:

$$\sum_{(\pi_0,\ldots,\pi_s)\in S(\pi)} (-1)^s (\langle \pi_0, \pi_0\rangle + \cdots + \langle \pi_s, \pi_s\rangle)^k = 0$$

$$\sum_{(\pi_0,\ldots,\pi_s)\in S(\pi)} (-1)^s(|\pi_0| - l(\pi_0))^1(\langle\pi_0,\pi_0\rangle + \cdots + \langle\pi_s,\pi_s\rangle)^{k-1} = 0$$

$$\vdots$$

$$\sum_{(\pi_0,\ldots,\pi_s)\in S(\pi)} (-1)^s(|\pi_0| - l(\pi_0))^{k-1}(\langle\pi_0,\pi_0\rangle + \cdots + \langle\pi_s,\pi_s\rangle)^1 = 0$$

$$\sum_{(\pi_0,\ldots,\pi_s)\in S(\pi)} (-1)^s(|\pi_0| - l(\pi_0))^k = 0.$$

The first condition may be removed; it is trivially true, since adding an empty first partition to a partition tuple flips the sign $(-1)^s$ and does not change the inner product sum. For each of the remaining $k$ conditions, a tuple $(\pi_0,\ldots,\pi_s) \in S(\pi) \setminus S'(\pi)$ has contribution 0, since $|\pi_0| - l(\pi_0) = 0$. Therefore it suffices to consider only the tuples in $S'(\pi)$. This completes the proof. $\square$

Now

$$\sum_{(\pi_0,\ldots,\pi_s)\in S'(\pi)} (-1)^s(|\pi_0| - l(\pi_0))^i(\langle\pi_0,\pi_0\rangle + \cdots + \langle\pi_s,\pi_s\rangle)^{k-i}$$

is an element of $\mathbb{R}[a_0,\ldots,a_{n-1}]$ where $\pi = (a_0,\ldots,a_{n-1})$. For generating partition tuples and calculating inner products, it is assumed that $a_i > a_j$ whenever $i < j$. Thus if $n = 5$, $\pi_0 = (a_0, a_4, a_1)$ is not a valid subpartition, but $\pi_0 = (a_0, a_1, a_4)$ is valid, and $\langle\pi_0,\pi_0\rangle = a_0 + 3a_1 + 5a_4$.

**Proposition 16.** *Let $n$ and $i$ be integers with $0 < i < n$. The polynomial*

$$\sum_{(\pi_0,\ldots,\pi_s)\in S'(\pi)} (-1)^s(|\pi_0| - l(\pi_0))^i$$

*is zero.*

**Proposition 17.** *Let $n$ and $k$ be integers with $0 < k < n$. The polynomial*

$$\sum_{(\pi_0,\ldots,\pi_s)\in S'(\pi)} (-1)^s(|\pi_0| - l(\pi_0))(\langle\pi_0,\pi_0\rangle + \cdots + \langle\pi_s,\pi_s\rangle)^{k-1}$$

*is zero if and only if the polynomial*

$$\sum_{\substack{(\pi_0,\ldots,\pi_s)\in S'(\pi)\\ a_0\in\pi_0}} (-1)^s(\langle\pi_0,\pi_0\rangle + \cdots + \langle\pi_s,\pi_s\rangle)^{k-1}$$

*is zero.*

**Proposition 18.** *Let $n$ and $k$ be integers with $0 < k < n$. If the polynomial*

$$\sum_{\substack{(\pi_0,\ldots,\pi_s)\in S'(\pi)\\ a_0\in\pi_0}} (-1)^s(\langle\pi_0,\pi_0\rangle + \cdots + \langle\pi_s,\pi_s\rangle)^{k-i}$$

*is zero for all $i$ such that $0 < i \leq k$, then the polynomial*

$$\sum_{(\pi_0,\ldots,\pi_s)\in S'(\pi)} (-1)^s(|\pi_0| - l(\pi_0))^i(\langle\pi_0,\pi_0\rangle + \cdots + \langle\pi_s,\pi_s\rangle)^{k-i}$$

*is zero for all $i$ such that $0 < i \leq k$.*

Now let $\pi = (a_0,\ldots,a_{n-1})$. Note that the coefficient of $a_i$ in $\langle\pi,\pi\rangle$ is

$$1 + \sum_{\substack{a_j\in\pi\\ j<i}} 2.$$

We can generalize this definition of the inner product.

**Definition.** Let $n > 0$ and let $\pi = (a_0,\ldots,a_{n-1})$. Let $W$ be an $n\times n$ matrix of positive integers, and let $c$ be a positive integer. Then we define the $(W,b)$-*inner product* of $\pi$ to be

$$\langle\pi,\pi\rangle_{W,b} = \sum_{a_i\in\pi} a_i \left(b + \sum_{a_j\in\pi} W_{ji}\right).$$

Note that $\langle\pi,\pi\rangle = \langle\pi,\pi\rangle_{Z,1}$ where $Z$ is the $n\times n$ matrix with twos above the diagonal and zeros everywhere else.

**Definition.** Let $n$ be a positive integer, and let $i_0, i_1, \ldots, i_k$ be integers such that $0 \leq i_0 < i_1 < \cdots < i_k < n$. For some nonnegative integer $j$ with $j \leq k$, define $u = (i_1,\ldots,i_j)$ and $v = (i_{j+1},\ldots,i_k)$. Then define $T(n,i_0,u,v,a)$ to be the coefficient of $i_1\cdots i_k$, scaled down by $k!$, in the polynomial

$$\sum_{\substack{(\pi_0,\ldots,\pi_s)\in S'(\pi)\\ a_0\in\pi_0\\ s\equiv 0 \pmod 2}} (\langle\pi_0,\pi_0\rangle_{W,0} + \cdots + \langle\pi_s,\pi_s\rangle_{W,0})^k$$

where $W$ is the $n\times n$ matrix with entries $W_{i_0 i_l} = a$ for $j < l \leq k$ and ones everywhere else.

Define $T'(n,i_0,u,v,a)$ identically except with $s \equiv 1 \pmod 2$.

**Proposition 19.** *Both $T(n,i_0,u,v,a)$ and $T'(n,i_0,u,v,a)$ are equal to*

$$F(1,n-k-1)i_1 i_2\cdots i_j(i_{j+1}+a-1)\cdots(i_k+a-1).$$

*Proof.* We show the proof for $T$, but it is almost identical for $T'$—identical, in fact, for all steps except the $k = 0$ case.

First we induct on $k$. Suppose $k = 0$. Then $T(n,i_0,u,v,a)$ is simply the number of partition tuples of $\pi = (a_0,\ldots,a_{n-1})$ of even length with $a_0$ contained in $\pi_0$. This is equal to the number of partition tuples of $(a_1,\ldots,a_{n-1})$, since for each of the latter set we can construct an even-length tuple of $\pi$ in exactly one way: if the original tuple has even length, we include $a_0$ at the beginning of $\pi_0$, and otherwise we add a new partition to the tuple containing only the element $a_0$. But the number of partition tuples of $(a_1, \ldots, a_{n-1})$ is $F(1, n-1)$ as desired.

Now let $k > 0$ and suppose that the statement is true for all smaller values. To prove that the statement is true for $k$, we induct on $|v|$. Suppose $|v| = 0$. Then $T(n, i_0, u, v, a)$ is the coefficient of $i_1 \cdots i_k$, scaled down by $k!$, in the polynomial

$$\sum_{\substack{(\pi_0,\ldots,\pi_s)\in S'(\pi)\\ a_0\in\pi_0\\ s\equiv 0 \pmod 2}} (\langle \pi_0, \pi_0\rangle_{W,0} + \cdots + \langle \pi_s, \pi_s\rangle_{W,0})^k$$

where $W$ is the $n \times n$ matrix with ones above the diagonal and zeros everywhere else. Let $u' = (i_1 - 1, i_2, \ldots, i_k)$. Let $u''$ be empty, and let $v'' = (i_2 - 1, \ldots, i_k - 1)$. Then

$$T(n, i_0, u, v, a) - T(n, i_0, u', v, a) = T(n-1, i_1, u'', v'', 2).$$

This arises from the fact that in any partition tuple where $a_{i_1}$ and $a_{i_1-1}$ are in different partitions, the contribution of this tuple to $T(n, i_0, u, v, a)$ is cancelled out by the contribution of the tuple to $T(n, i_0, u', v, a)$ when $a_{i_1}$ and $a_{i_1-1}$ are swapped. Therefore in the remaining tuples, $a_{i_1}$ and $a_{i_1-1}$ are in the same partition, and as they are adjacent, the difference in their coefficients is 1. Hence the tuple's difference in contributions is the product of the coefficients of the remaining variables: $a_{i_2}, \ldots, a_{i_k}$. Grouping $a_{i_1}$ and $a_{i_1-1}$ as one variable with a weight of 2 on the other variables, we can establish a bijection with $T(n-1, i_1, u'', v'', 2)$. But then by the inductive hypothesis

$$\begin{aligned}
T(n, i_0, u, v, a) &= T(n, i_0, u', v, a) + T(n-1, i_1, u'', v'', 2)\\
&= F(1, n-k-1)(i_1 - 1)i_2 \cdots i_k + F(1, n-k-1)(i_2 - 1 + 2 - 1) \cdots (i_k - 1 + 2 - 1)\\
&= F(1, n-k-1) i_1 i_2 \cdots i_k.
\end{aligned}$$

Now let $|v| > 0$ and suppose that the statement is true for all smaller sets. We have

$$T(n, i_0, u, v, a) - T(n, i_0, u \cup \{i_{j+1}\}, v \setminus \{i_{j+1}\}, a)$$

is equal to $(a-1)$ times the portion of

$$T(n, i_0, u, v \setminus \{i_{j+1}\}, a)$$

contributed by tuples where $i_0$ is in the same partition as $i_{j+1}$. This is because changing $w_{i_0 i_{j+1}}$ from $a$ to 1 affects only tuples with $i_0$ and $i_j$ in the same partition, and the resulting contribution is $(a-1)$ times the coefficients of the remainder of $u$ and $v$. But the portion of

$$T(n, i_0, u, v \setminus \{i_{j+1}\}, a)$$

with $i_0$ and $i_{j+1}$ in the same partition is exactly

$$T(n-1, i_0, u, v', a+1)$$

where $v' = (i_{j+2} - 1, \ldots, i_k - 1)$. This is obtained by assimilating the variable $i_{j+1}$ into $i_0$, which increases the weight by 1 but decrements all indices after $j + 1$. Hence, by the inductive hypothesis,

$$\begin{aligned} T(n, i_0, u, v, a) &= T(n, i_0, u \cup \{i_{j+1}\}, v \setminus \{i_{j+1}\}, a) + (a-1)T(n-1, i_0, u, v', a+1) \\ &= F(1, n-k-1) i_1 i_2 \cdots i_{j+1} (i_{j+2} + a - 1) \cdots (i_k + a - 1) \\ &\quad + F(n-k-1)(a-1) i_1 i_2 \cdots i_j (i_{j+2} + a - 1) \cdots (i_k + a - 1) \\ &= F(n-k-1) i_1 i_2 \cdots i_j (i_{j+1} + a - 1)(i_{j+2} + a - 1) \cdots (i_k + a - 1). \end{aligned}$$

This completes the induction. $\square$

For the general case, we must generalize our definition of $T$. Let $W$ be an upper triangular $n \times n$ matrix of positive integers. Let $k \geq 0$, let $i$ be a vector of $k$ nonnegative integers, with $i_1 < i_2 < \cdots < i_k < n$, and let $e$ be a vector of $k$ positive integers. Then let $[a_{i_1}^{e_1} \cdots a_{i_k}^{e_k}]T(W)$ be the coefficient of

$$\prod_{j=1}^{k} a_{i_j}^{e_j}$$

in

$$\sum_{\substack{(\pi_0, \ldots, \pi_s) \in S'(\pi) \\ a_0 \in \pi_0}} (-1)^s \left( \langle \pi_0, \pi_0 \rangle_W + \cdots + \langle \pi_s, \pi_s \rangle_W \right)^{e_1 + \cdots + e_k}.$$

From here on, we will assume that this is divided by the number of distinct permutations of $a_{i_1}^{e_1} \cdots a_{i_k}^{e_k}$, so that we are in fact counting the cases when $a_{i_1}$ is chosen from the first factor, $a_{i_k}$ from the last factor, and so forth.

Similarly, for any $j_1$ and $j_2$ with $0 \leq j_1 < j_2 < n$, let $[a_{i_1}^{e_1} \cdots a_{i_k}^{e_k}]T_{j_1, j_2}(W)$ be equal to $[a_{i_1}^{e_1} \cdots a_{i_k}^{e_k}]T(W)$ restricted to the partition tuples where $a_{j_1}$ and $a_{j_2}$ are in the same partition.

**Proposition 20.** *For all valid parameters where $W$ is strictly upper triangular,*

$$[a_{i_1}^{e_1} \cdots a_{i_k}^{e_k}]T(W) = 0.$$

*Proof.* We induct on the index-exponent sum $\sum_{j=1}^{k} i_j + e_j$. If the sum is 0, then $[a_{i_1}^{e_1} \cdots a_{i_k}^{e_k}]T(W)$ is simply the value

$$\sum_{\substack{(\pi_0, \ldots, \pi_s) \in S'(\pi) \\ a_0 \in \pi_0}} (-1)^s$$

which is 0.

Now suppose the index-exponent sum is positive. Then $k \geq 1$. If $i_1 = 0$, then $[a_{i_1}^{e_1} \cdots a_{i_k}^{e_k}]T(W)$ is trivially 0, since for any partition tuple the coefficient of $a_0$ within its inner product is $W_{0,0} = 0$.

If $i_1 > 0$, note that most contributions in $[a_{i_1}^{e_1} \cdots a_{i_k}^{e_k}]T(W)$ can be placed in bijection with equal contributions from $[a_{i_1-1}^{e_1} \cdots a_{i_k}^{e_k}]T(W')$, where $W'$ is the matrix obtained from $W$ by swapping all corresponding entries of rows $i_1 - 1$ and $i_1$ except for entry $W_{i_1-1,i_1}$. In particular, let $\alpha = (\pi_0, \dots, \pi_s)$ be a partition tuple with $a_{i_1}$ and $a_{i_1-1}$ in different partitions. Let $\alpha'$ be the tuple constructed from $\alpha$ by swapping $a_{i_1}$ and $a_{i_1-1}$. Then $\alpha$ has the same contribution in $[a_{i_1}^{e_1} \cdots a_{i_k}^{e_k}]T(W)$ as $\alpha'$ has in $[a_{i_1-1}^{e_1} \cdots a_{i_k}^{e_k}]T(W')$.

Hence, the only tuples with a nonzero contribution to

$$[a_{i_1}^{e_1} \cdots a_{i_k}^{e_k}]T(W) - [a_{i_1-1}^{e_1} \cdots a_{i_k}^{e_k}]T(W')$$

are those tuples where $a_{i_1}$ and $a_{i_1-1}$ are in the same partition. Let $\alpha$ be such a tuple. The last $e_2 + \cdots + e_k$ factors in the power can be factored out of the difference, since we are looking at the coefficients of the same monomial $a_{i_2}^{e_2} \cdots a_{i_k}^{e_k}$, and although the weights of $i_1$ and $i_1 - 1$ have been swapped, this makes no difference since they are both in the same partition. Then if $x$ is the coefficient of $a_{i_1}$ in the inner product sum, the difference is

$$x^{e_1} - (x - W_{i_1-1,i_1})^{e_1} = \sum_{j=0}^{e_1-1} \binom{e_1}{j} (-W_{i_1-1,i_1})^{e_1-1-j} x^j$$

times the coefficients due to the remaining variables $a_{i_2}, \dots, a_{i_k}$. But for each $j$, the product of $x^j$ and the coefficients of the remaining variables, is exactly the coefficient of $a_{i_1}^j a_{i_2}^{e_2} \dots a_{i_k}^{e_k}$. Hence,

$$\begin{aligned} &[a_{i_1}^{e_1} \cdots a_{i_k}^{e_k}]T(W) - [a_{i_1-1}^{e_1} \cdots a_{i_k}^{e_k}]T(W') \\ &= \sum_{j=0}^{e_1-1} \binom{e_1}{j} (-W_{i_1-1,i_1})^{e_1-1-j} [a_{i_1}^j a_{i_2}^{e_2} \cdots a_{i_k}^{e_k}] T_{i_1-1,i_1}(W). \end{aligned}$$

By the inductive hypothesis, $[a_{i_1-1}^{e_1} \cdots a_{i_k}^{e_k}]T(W') = 0$. For the remaining terms, note that

$$[a_{i_1}^{e_1} \cdots a_{i_k}^{e_k}]T_{i_1-1,i_1}(W) = [a_{i_1-1}^{e_1} \cdots a_{i_k-1}^{e_k}]T(W')$$

where $W'$ is the $n-1 \times n-1$ matrix constructed from $W$ by adding $W_{i_1,i_j}$ into $W_{i_1-1,i_j}$ for each $j$, and adding $W_{i_1-1,i_1}$ into $W_{i_1-1,i_1-1}$, and then removing row $i_1$ and column $i_1$. Essentially we are removing variable $i_1 - 1$, but so as to avoid affecting the coefficients of other variables, we must somehow keep the weights of $i_1 - 1$ on other variables. Since variable $i_1$ is always in the same partition, it suffices to simply add each weight of $i_1 - 1$ into the corresponding weight of $i_1$. But now the entry $W_{i_1-1,i_1-1}$ is nonzero, so the matrix $W$ is not strictly upper triangular. This may be fixed by applying the binomial theorem again; we have

$$[a_{i_1-1}^{e_1} \cdots a_{i_k-1}^{e_k}]T(W') = \sum_{j=0}^{e_1} \binom{e_1}{j} W_{i_1-1,i_1}^{e_1-1-j} [a_{i_1-1}^j \cdots a_{i_k-1}^{e_k}]T(W'')$$

where $W''$ is obtained from $W'$ by setting entry $W'_{i_1-1,i_1-1}$ to 0. But $W''$ is strictly upper triangular, so by the inductive hypothesis, this sum is 0. It follows that

$$[a_{i_1}^{e_1} \cdots a_{i_k}^{e_k}]T(W) = 0.$$

$\square$

# 6 Acknowledgements

I thank my mentor, Mr. Vishal Arul, for his guidance and patience during my first foray into math research. I also thank Dr. Tristan Bozec for suggesting this problem, and Dr. Tanya Khovanova for her guidance on writing quality papers. In addition, I express gratitude to Dr. Slava Gerovitch and Dr. Pavel Etingof for organizing the program. I am grateful to MIT-PRIMES for giving me the opportunity to undertake this research.

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