Generalizations of Hall-Littlewood Polynomials

Mihir Singhal, Christopher Ryba

Abstract

Hall-Littlewood polynomials are important functions in various fields of mathematics and quantum physics, and can be defined combinatorially using a model of path ensembles. Wheeler and Zinn-Justin applied a reflection construction to this model to obtain an expression for type BC Hall-Littlewood polynomials. Borodin applied a single-parameter deformation to the model and obtained a formula for generalized Hall-Littlewood polynomials. Borodin has asked whether a similar generalization could be applied to type BC Hall-Littlewood polynomials. We present the model incorporating Borodin's generalization. We also obtain expressions for polynomials that were previously studied by Borodin, in addition to an expression for generalized type BC Hall-Littlewood polynomials.

1 Introduction

Both type A and type BC Hall-Littlewood polynomials are useful for modeling systems of particles in quantum physics. Since these models are idealized and real systems have a lot of parameters that can affect the systems, generalizing the model with additional parameters can help to more accurately describe physical systems.

Hall-Littlewood polynomials (also referred to as type A Hall-Littlewood polynomials) are symmetric functions that are a single-parameter deformation of the Schur polynomials and are given by the following expression [1] (where the elements of S_n act by permuting the variables x_i in the obvious way):

$$P_\lambda(x_1,\dots,x_n;t) = \frac{1}{v_\lambda(t)} \sum_{\sigma\in S_n} \sigma\left(\prod_{1\le i\le n} x_i^{\lambda_i} \prod_{1\le i<j\le n} \frac{x_i - t x_j}{x_i - x_j} \right).$$

Wheeler and Zinn-Justin [2] constructed hyperoctahedrally symmetric type BC Hall-Littlewood polynomials by modifying a statistical mechanical model that generated the aforementioned type A Hall-Littlewood polynomials and obtained a symmetrization presentation (where the elements of $H_n \cong S_n \ltimes \{\pm 1\}^n$ act by permuting and inverting the variables x_i). These functions are Laurent polynomials and have two additional parameters, γ and δ. They are given by the following expression:

$$K_\lambda(x_1^{\pm1},\dots,x_n^{\pm1};t;\gamma,\delta) = \\ \frac{1}{v_\lambda(t)} \sum_{\omega\in H_n} \omega\left(\prod_{1\le i\le n} \left(x_i^{\lambda_i} \frac{(1-\gamma\bar{x}_i)(1-\delta\bar{x}_i)}{1-\bar{x}_i^2} \right) \prod_{1\le i<j\le n} \frac{(x_i - t x_j)(1 - t\bar{x}_i\bar{x}_j)}{(x_i - x_j)(1-\bar{x}_i\bar{x}_j)} \right). \tag{1}$$

Separately, Borodin [3] generalized the model by adding an additional parameter s. Wheeler and Zinn-Justin, in [2], mention this generalization but do not incorporate it into their results. We incorporate the generalization in [3] to the model and framework in [2] and obtain the same polynomial as in [3], up to a constant factor, to generalize type A Hall-Littlewood polynomials:

$$F_\lambda(x_1,\dots,x_n;t,s) = \frac{(s^2;t)_\lambda}{v_\lambda(t)} \cdot \frac{1}{\prod_{1\le i\le n}(1-s x_i)} \sum_{\sigma\in S_n} \sigma\left(\prod_{1\le i\le n} \left(\frac{x_i - s}{1 - s x_i}\right)^{\lambda_i} \prod_{1\le i<j\le n} \frac{x_i - t x_j}{x_i - x_j} \right).$$

We also apply the generalized weights in [3] to the type BC polynomials in (1) to obtain our main result, a family of symmetric polynomials that we call generalized type BC polynomials:

$$H_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; t, s; \gamma, \delta) = \\ \frac{(s^2;t)_\lambda}{v_\lambda(t)} \sum_{\omega \in H_n} \omega \left(\prod_{1 \le i \le n} \left(\left(\frac{x_i - s}{1 - s x_i} \right)^{\lambda_i} \frac{(1-\gamma \bar{x}_i)(1-\delta \bar{x}_i)}{(1-\bar{x}_i^2)(1-s x_i)} \right) \prod_{1 \le i < j \le n} \frac{(x_i - t x_j)(1 - t \bar{x}_i \bar{x}_j)}{(x_i - x_j)(1 - \bar{x}_i \bar{x}_j)} \right).$$

2 Definitions and Notation

2.1 Integrable model, basic definitions

We set up an integrable model as in [2], with an infinite dimensional vector space V representing occupation numbers of particles:

$$V = \mathrm{Span}\{|m_0\rangle_0 \otimes |m_1\rangle_1 \otimes |m_2\rangle_2 \otimes \ldots\},$$

where $m_i \ge 0$ for all $i \ge 0$ and only finitely many of the m_i are nonzero. Also, we let V_i denote $\mathrm{Span}\{|m_i\rangle_i\}$, so that $V = V_0 \otimes V_1 \otimes V_2 \otimes \ldots$. We also define $|\lambda\rangle$ for a partition λ:

$$|\lambda\rangle = |m_0(\lambda)\rangle_0 \otimes |m_1(\lambda)\rangle_1 \otimes |m_2(\lambda)\rangle_2 \otimes \cdots \in V,$$

where $m_i(\lambda)$ denotes the number of instances of i in λ. Note that these form a basis of V; let $\langle\lambda| \in V^*$ denote the corresponding element of $|\lambda\rangle$ in the dual basis of the basis of partitions.

We define $\bar{x}$ to be $\frac{1}{x}$. We also define the standard q-Pochhammer symbol:

$$(a;q)_n = (1-a)(1-qa)\ldots(1-aq^{n-1}),$$

with the right hand side vacuously equal to 1 when $n = 0$. Moreover, we extend this symbol to partitions:

$$(a;q)_\lambda = \prod_{i \in \lambda} (a;q)_i,$$

where the product is taken with multiplicity.

Furthermore, we define the function $v_\lambda(t)$ as in [1]:

$$v_\lambda(t) = \prod_{i=0}^{\infty} \left(\prod_{j=1}^{m_i(\lambda)} \frac{1-t^j}{1-t} \right).$$

2.2 Regular and normalized R matrix, Yang-Baxter equation, and unitarity relation

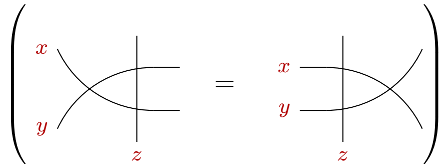
We define the R matrix as in [2], which operates on the auxiliary two-dimensional vector spaces W_a and W_b.

$$R_{ab}(x/y) = \begin{pmatrix} \frac{1-tz}{1-z} & 0 & 0 & 0 \\ 0 & t & \frac{(1-t)z}{1-z} & 0 \\ 0 & \frac{1-t}{1-z} & 1 & 0 \\ 0 & 0 & 0 & \frac{1-tz}{1-z} \end{pmatrix}_{ab},$$

where $z = x/y$. Here the subscripts a and b indicate that the R matrix operates on $W_a \otimes W_b$. The two elements of the bases of these auxiliary spaces are represented by the symbols $\circ$ and $\bullet$ (so the R matrix operates on the basis $(\circ \otimes \circ, \circ \otimes \bullet, \bullet \otimes \circ, \bullet \otimes \bullet))$. These symbols represent high-spin and low-spin particles, respectively. This matrix is represented graphically as a crossing of two lines, as in equation (10) of [2].

The R matrix satisfies the Yang-Baxter equation [2]:

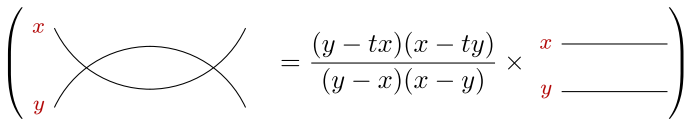
$$R_{ab}(x/y)R_{ac}(x/z)R_{bc}(y/z) = R_{bc}(y/z)R_{ac}(x/z)R_{ab}(x/y).$$



In these diagrams, a crossing of the lines represents an application of the R matrix, and each line is associated with a different auxiliary space. The variable labeling each line represents the argument used for the R matrix.

The R matrix also satisfies the unitarity relation [2]:

$$R_{ab}(x/y)R_{ba}(y/x) = \frac{(y-tx)(x-ty)}{(y-x)(x-y)}.$$



We also define $\mathcal{R}$, the normalized R matrix, to be a multiple of the R matrix such that the unitarity relation has right hand side 1:

$$\mathcal{R}_{ab}(x/y) = \frac{1-z}{1-tz}R_{ab}(x/y),$$

so that

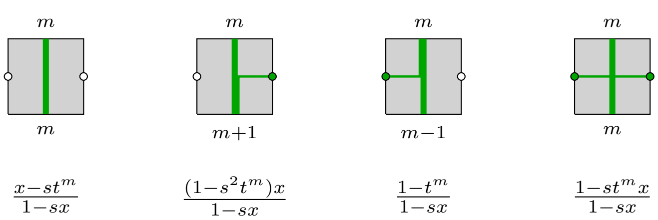
$$\mathcal{R}_{ab}(x/y)\mathcal{R}_{ba}(y/x) = 1.$$

2.3 L operator, intertwining equations

The integrable model in [2] is based on the L operator, an element of $\mathrm{End}(W_a \otimes V)$, where W_a is a two-dimensional auxiliary space. We define a generalized version of this L operator with weights as in [3], defining $L_{a,i}(x) \in \mathrm{End}(W_a \otimes V_i)$ by its operation on the basis elements of V_i as follows:

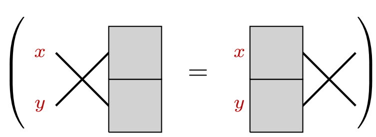
$$L_{a,i}(x)\left|m\right\rangle_i = \frac{1}{1-sx}\begin{pmatrix} (x-st^m)\left|m\right\rangle_i & (1-s^2t^m)x\left|m+1\right\rangle_i \\ (1-t^m)\left|m-1\right\rangle_i & (1-st^mx)\left|m\right\rangle_i \end{pmatrix}_a,$$

where the 2 by 2 matrix acts on W_a. The components of this operator are graphically represented as follows:



In the graphical representation of an L operator (a "tile"), the number of lines entering at the top corresponds with the basis element of V_i on which it is acting, and the number of lines exiting the bottom corresponds with the basis element of its output. By Proposition 2.5 of [3], we have an intertwining equation analogous to equation (14) of [2]:

$$R_{ab}(x/y)L_{a,i}(x)L_{b,i}(y) = L_{b,i}(y)L_{a,i}(x)R_{ab}(x/y).$$



We also define a dual operator L^*:

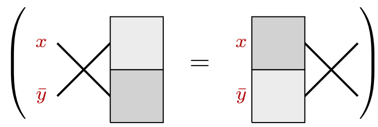
$$L^*_{a,i}(x) = \frac{x-s}{1-sx} L_{a,i}(\bar{x}),$$

so that:

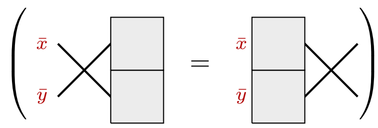
$$L^*_{a,i}(x)\left|m\right\rangle_i = \frac{1}{1-sx}\begin{pmatrix}(1-st^m x)\left|m\right\rangle_i & (1-s^2t^m)\left|m+1\right\rangle_i \\ (1-t^m)x\left|m-1\right\rangle_i & (x-st^m)\left|m\right\rangle_i\end{pmatrix}_a.$$

The following intertwining equations also hold:

$$R_{ab}(xy)L_{a,i}(x)L^*_{b,i}(y) = L^*_{b,i}(y)L_{a,i}(x)R_{ab}(xy),$$



$$R_{ab}(y/x)L^*_{a,i}(x)L^*_{b,i}(y) = L^*_{b,i}(y)L^*_{a,i}(x)R_{ab}(y/x).$$



2.4 Permutation $\mathcal{R}$ matrix, D matrix

We define the permutation $\mathcal{R}$ matrix, $\mathcal{R}^{\rho}_{\sigma}$, as in Appendix A of [2]. In other words, if σ and ρ are permutations, then decompose $\sigma\rho^{-1}$ into a series of transpositions, and let $\mathcal{R}^{\rho}_{\sigma}$ be the product of the corresponding R matrices. If ρ is the identity permutation, we abbreviate $\mathcal{R}^{\rho}_{\sigma}$ by simply $\mathcal{R}_{\sigma}$.

Similarly, we define the D matrix $D_{1\ldots n}$ identically to the F matrix in Appendix A of [2]. (We have changed the variable name because of a conflict with F functions.) The D matrix, $D_{1\ldots n} \in \mathrm{End}(W_1 \otimes \ldots \otimes W_n)$, is defined so that it satisfies:

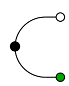
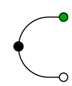
$$F_{\sigma(1)\ldots\sigma(n)}\mathcal{R}^{1\ldots n}_{\sigma} = F_{1\ldots n}.$$

2.5 Boundary covector, boundary B operator, reflection and fish equations

We define the boundary covector $\left\langle K\right|_{a\bar{a}} \in W^*_a \otimes W^*_{\bar{a}}$ identically to section 2.7 of [2]:

$$\left\langle K\right|_{a\bar{a}} = \begin{pmatrix}1 & 0\end{pmatrix}_a \otimes \begin{pmatrix}0 & 1\end{pmatrix}_{\bar{a}} - t\begin{pmatrix}0 & 1\end{pmatrix}_a \otimes \begin{pmatrix}1 & 0\end{pmatrix}_{\bar{a}}.$$

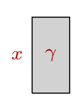
The boundary covector is represented graphically by a U-turn vertex as in the following equation. When interpreted as a four-dimensional vector, it has two nonzero components:

$$\left\langle K\right|_{a\bar{a}}\begin{pmatrix}1\\0\end{pmatrix}_a \otimes \begin{pmatrix}0\\1\end{pmatrix}_{\bar{a}} =$$  $$= 1, \qquad \left\langle K\right|_{a\bar{a}}\begin{pmatrix}0\\1\end{pmatrix}_a \otimes \begin{pmatrix}1\\0\end{pmatrix}_{\bar{a}} =$$  $$= -t.$$

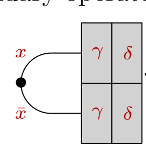
We also define the boundary B operator $B^{(\gamma)}_{a,i}(x) \in \mathrm{End}(W_a \otimes V_i)$ as in [2]. Letting m be the number of particles at the top of the tile, we define the boundary operator by its action on the basis element $\left|m\right\rangle_i$ of V_i:

$$B^{(\gamma)}_a(x)|m\rangle_i = \begin{pmatrix}t^m\left|m\right\rangle_i & \gamma x t^m\left|m+1\right\rangle_i \\ (1-t^m)\left|m-1\right\rangle_i & (1-\gamma x t^m)\left|m\right\rangle_i\end{pmatrix}_a.$$

The graphical representation of this boundary operator is a tile with the parameter γ inside it:



Additionally, we define a combination of the boundary covector and the boundary operator:

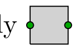
$$\left\langle K(x;\gamma,\delta)\right|_{a\bar{a}} = \left\langle K\right|_{a\bar{a}} B^{(\gamma)}_{\bar{a},-2}(\bar{x})B^{(\delta)}_{\bar{a},-1}(\bar{x})B^{(\gamma)}_{a,-2}(x)B^{(\delta)}_{a,-1}(x) =$$ 

The above expression is an operator in $W^*_a \otimes W^*_{\bar{a}} \otimes \mathrm{End}(\widehat{V})$, where $\widehat{V} = V_{-2} \otimes V_{-1}$.

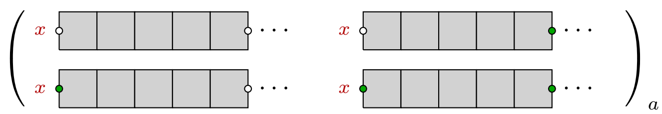
2.6 Single- and double-row transfer matrices

We also define row transfer matrices as in sections 2.6 and 2.9 of [2]. The single-row transfer matrix $T_a(x) \in \mathrm{End}(W_a \otimes V)$ is defined by:

$$T_a(x) = \prod_{i=0}^{\infty} L_{a,i}(x) =$$

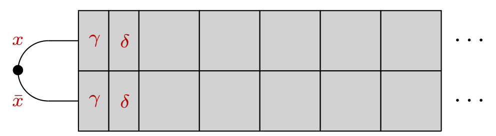
We assume that $\left|\frac{x-s}{1-sx}\right| < 1$ so that the product converges to 0 unless there are only  tiles sufficiently far to the right. Recall that here W_a is a 2-dimensional auxiliary space and $V = V_0 \otimes V_1 \otimes V_2 \otimes \ldots$. If we treat $T_a(x)$ as a 2 by 2 matrix over W_a with components in $\mathrm{End}(V)$, then it only has two nonzero components:

$$T_a(x) = \begin{pmatrix} 0 & T_+(x) \\ 0 & T_-(x) \end{pmatrix}_a =$$



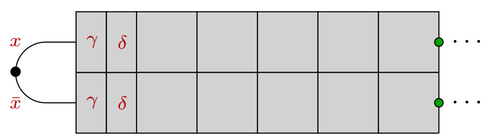
Similarly, we define the double-row transfer matrix $\mathbb{T}_{a\bar{a}}(x;\gamma,\delta) \in W_a^* \otimes W_{\bar{a}}^* \otimes \mathrm{End}(\widehat{V} \otimes V)$ as in [2]:

$$\mathbb{T}_{a\bar{a}}(x;\gamma,\delta) = \langle K(x;\gamma,\delta)|_{a\bar{a}} \prod_{i=0}^{\infty} L_{a,i}(\bar{x}) \prod_{j=0}^{\infty} L_{a,j}(x) =$$



We consider its only nonzero component:

$$\mathbb{T}_{--}(x;\gamma,\delta) =$$



3 Results

In this section, we derive the F function of [3] in terms of our notation. (There is a difference of a constant factor, but it is not important.)

3.1 Definition of F function

We define the F function as follows:

$$\left(\prod_{1 \le i \le n} x_i\right) F_\lambda(x_1,\ldots,x_n;t,s) = \langle\lambda|\, T_+(x_n)\ldots T_+(x_1)\,|0\rangle$$

3.2 Expression for F functions

We now obtain the symmetrization representation for F functions of [3] (a family of symmetric polynomials indexed by partitions) using the techniques of [2]. Define the column transfer matrix, an element of $\mathrm{End}(W_1 \otimes \ldots \otimes W_n \otimes V_i)$:

$$S^{[i]}(x_1,\ldots,x_n) = L_{n,i}(x_n)\ldots L_{1,i}(x_1) \in \mathrm{End}(W_1 \otimes \cdots \otimes W_n \otimes V_i).$$

We also define the components of this matrix in V_i (note that these components are independent of i):

$$S^{[l,m]}(x_1,\ldots,x_n) = \langle l|_i\, S^{[i]}(x_1,\ldots,x_n)\,|m\rangle_i \in \mathrm{End}(W_1 \otimes \cdots \otimes W_n).$$

We also define a twisted column transfer matrix:

$$\tilde{S}^{[i]}_{1\ldots n}(x_1,\ldots,x_n) = D_{1\ldots n}(x_1,\ldots,x_n) S^{[i]}_{1\ldots n}(x_1,\ldots,x_n) D^{-1}_{1\ldots n}(x_1,\ldots,x_n).$$

This twisted column transfer matrix, by the results of Appendix A of [2], is independent of its permutation. Now we obtain an expression for the twisted column transfer matrix:

Theorem 1. *If $0 \le m \le n$, then*

$$\tilde{S}^{[m,0]}x_1,\ldots,x_n = (s^2;t)_m \sum_{\substack{I\in\{1,\ldots,n\}\\ |I|=m}} \bigotimes_{i\in I} \begin{pmatrix} 0 & \frac{x_i}{1-sx_i} \\ 0 & 0 \end{pmatrix}_i \bigotimes_{j\notin I} \begin{pmatrix} \frac{x_j-s}{1-sx_j}\prod_{k\in I}\frac{x_j-tx_k}{x_j-x_k} & 0 \\ 0 & 1 \end{pmatrix}_j \tag{2}$$

.

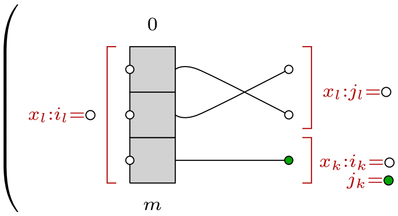
Proof. The proof is almost identical to that of Theorem 2 of [2]; we replicate it here. As in (43) of [2], we have:

$$\left[\widetilde{S}^{[m,0]}_{1\ldots n}(x_1,\ldots,x_n)\right]^{j_1\ldots j_n}_{i_1\ldots i_n} = \left[S^{[m,0]}_{\sigma(1)\ldots\sigma(n)}\left(x_{\sigma(1)},\ldots,x_{\sigma(n)}\right)\mathcal{R}^{1\ldots n}_{\sigma}\mathcal{R}^{\rho}_{1\ldots n}\right]^{j_1\ldots j_n}_{i_1\ldots i_n} \prod_{1\le k<l\le n} b^{-1}_{j_k,j_l}(x_k,x_l), \tag{3}$$

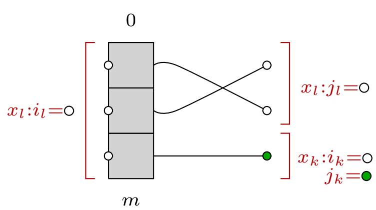
where $i_{\sigma(1)} \ge \cdots \ge i_{\sigma(n)}$ and $j_{\rho(1)} \le \cdots \le j_{\rho(n)}$. Furthermore, from pg. 14 of [2],

$$\left[S^{[m,0]}_{\sigma(1)\ldots\sigma(n)}\left(x_{\sigma(1)},\ldots,x_{\sigma(n)}\right)\mathcal{R}^{1\ldots n}_{\sigma}\mathcal{R}^{\rho}_{1\ldots n}\right]^{j_1\ldots j_n}_{i_1\ldots i_n} =$$

$$\delta_{0,\#\{k:i_k=\bullet,j_k=\circ\}} \prod_{k:i_k=\bullet}\prod_{l:j_l=\circ}\frac{(x_l-x_k)}{(x_l-tx_k)} \times \Bigg(\quad \Bigg). \tag{4}$$



Furthermore, similarly to (46) of [2],



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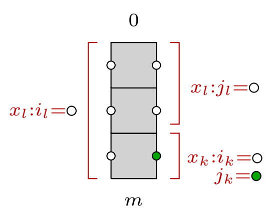


$$= \delta_{m,\#\{k:i_k=\circ,j_k=\bullet\}}(s^2;t)_m \prod_{l:\left\{\substack{i_l=\circ\\ j_l=\circ}\right.}\frac{x_l-s}{1-sx_l}\prod_{k:\left\{\substack{i_k=\circ\\ j_k=\bullet}\right.}\frac{x_k}{1-sx_k}. \tag{5}$$

Note that we also have:

$$\prod_{1\le k<l\le n} b^{-1}_{j_k,j_l}(x_k,x_l) = \prod_{k:j_k=\bullet}\prod_{l:j_l=\circ}\frac{(x_l-tx_k)}{(x_l-x_k)}. \tag{6}$$

Now, combining (3), (4), (5), and (6), we get:

$$\left[\widetilde{S}^{[m,0]}_{1\ldots n}(x_1,\ldots,x_n)\right]^{j_1\ldots j_n}_{i_1\ldots i_n} =$$

$$\delta_{0,\#\{k:i_k=\bullet,j_k=\circ\}}\delta_{m,\#\{k:i_k=\circ,j_k=\bullet\}} \prod_{l:\left\{\substack{i_l=\circ\\ j_l=\circ}\right.}\frac{x_l-s}{1-sx_l}\prod_{k:\left\{\substack{i_k=\circ\\ j_k=\bullet}\right.}\frac{x_k}{1-sx_k}\prod_{k:\left\{\substack{i_k=\circ\\ j_k=\bullet}\right.}\prod_{l:j_l=\circ}\frac{(x_l-tx_k)}{(x_l-x_k)},$$

which immediately leads to the expression in (2). $\square$

We can now use the expression for twisted transfer matrices given by (2) similarly to Lemma 3 of [2] to recover an expression for F functions:

Theorem 2. *The F functions are given by:*

$$F_\lambda(x_1,\ldots,x_n;t,s) = \frac{(s^2;t)_\lambda}{v_\lambda(t)} \cdot \frac{1}{\prod_{1\le i\le n}(1-sx_i)} \sum_{\sigma\in S_n} \sigma\left(\prod_{1\le i\le n}\left(\frac{x_i-s}{1-sx_i}\right)^{\lambda_i} \prod_{1\le i<j\le n} \frac{x_i-tx_j}{x_i-x_j}\right). \tag{7}$$

Proof. We have, as a consequence of the graphical representation of $\left(\prod_{1\le i\le n} x_i\right) F_\lambda(x_1,\ldots,x_n;t,s)$, that

$$\left(\prod_{1\le i\le n} x_i\right) F_\lambda(x_1,\ldots,x_n;t,s) = \langle\circ,\ldots,\circ| \prod_{i=0}^{\infty} S^{[m_i(\lambda),0]}(x_1,\ldots,x_n)\,|\bullet,\ldots,\bullet\rangle\,. \tag{8}$$

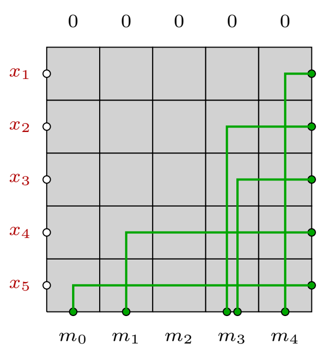
Now, since we also have

$$\langle\circ,\ldots,\circ|\,D_{1\ldots n} = \langle\circ,\ldots,\circ|\,, \qquad D_{1\ldots n}^{-1}\,|\bullet,\ldots,\bullet\rangle = |\bullet,\ldots,\bullet\rangle\,,$$

we can conjugate all the S matrices in (8) by $D_{1\ldots n}$ to obtain twisted S matrices instead:

$$\begin{aligned}\left(\prod_{1\le i\le n} x_i\right) F_\lambda(x_1,\ldots,x_n;t,s) &= \langle\circ,\ldots,\circ| \prod_{i=0}^{\infty} \widetilde{S}^{[m_i(\lambda),0]}(x_1,\ldots,x_n)\,|\bullet,\ldots,\bullet\rangle \\ &= \langle\circ,\ldots,\circ| \prod_{i=0}^{\lambda_1} \widetilde{S}^{[m_i(\lambda),0]}(x_1,\ldots,x_n)\,|\bullet,\ldots,\bullet\rangle\,. \end{aligned} \tag{9}$$

Now we can use (2) to compute (9) explicitly. The $\widetilde{S}$ operator only contains spin-raising ($|\bullet\rangle \to |\circ\rangle$) and not spin-lowering ($|\circ\rangle \to |\bullet\rangle$) operators, so we can conclude that the expression in (9) is equal to the symmetrization of any single term that it gives. The simplest such term is graphically represented (where each column is a twisted operator) as



$$= (s^2;t)_\lambda \prod_{1\le i\le n} \frac{x_i}{1-sx_i} \prod_{1\le i\le n}\left(\frac{x_i-s}{1-sx_i}\right)^{\lambda_i} \prod_{i,j:\lambda_i>\lambda_j} \frac{x_i-tx_j}{x_i-x_j},$$

where the diagram is for the specific case $\lambda = (4,3,3,1,0)$ and the equality is obtained using (2). Thus we have:

$$\left(\prod_{1\le i\le n} x_i\right) F_\lambda(x_1,\ldots,x_n;t,s) = (s^2;t)_\lambda \sum_{\sigma\in S_n/S_n^\lambda} \sigma\left(\prod_{1\le i\le n} \frac{x_i}{1-sx_i} \prod_{1\le i\le n}\left(\frac{x_i-s}{1-sx_i}\right)^{\lambda_i} \prod_{i,j:\lambda_i>\lambda_j} \frac{x_i-tx_j}{x_i-x_j}\right).$$

Canceling a factor of $\prod_{1\le i\le n} x_i$ from both sides and using the results of Section 1, Chapter III of [1], we obtain (7) as desired. $\square$

3.3 Definition of and expression for H functions

We define the H functions to generalize type BC functions in a similar way:

$$\left(\prod_{1\leq i\leq n} x_i\right) H_\lambda(x_1,\ldots,x_n;t,s;\gamma,\delta) = \langle\widehat{0}|\otimes\langle\lambda|\,\mathbb{T}_{--}(x_n;\gamma,\delta)\ldots\mathbb{T}_{--}(x_1;\gamma,\delta)\,|0\rangle\otimes|\widehat{0}\rangle.$$

The double-row column transfer matrices are defined as follows:

$$\begin{aligned}\mathbb{S}^{[i]}(x_1^{\pm1},\ldots,x_n^{\pm1}) &= L_{\bar{n},i}(\bar{x}_n)L_{n,i}(x_n)\ldots L_{\bar{1},i}(\bar{x}_1)L_{1,i}(x_1)\\ &\in \mathrm{End}(W_{\bar{1}}\otimes W_1\otimes\cdots\otimes W_{\bar{n}}\otimes W_n\otimes V_i),\\ \mathbb{B}^{[i]}(x_1^{\pm1},\ldots,x_n^{\pm1};\gamma) &= B^{(\gamma)}_{\bar{n},i}(\bar{x}_n)B^{(\gamma)}_{n,i}(x_n)\ldots B^{(\gamma)}_{\bar{1},i}(\bar{x}_1)B^{(\gamma)}_{1,i}(x_1)\\ &\in \mathrm{End}(W_{\bar{1}}\otimes W_1\otimes\cdots\otimes W_{\bar{n}}\otimes W_n\otimes V_i).\end{aligned}$$

Their components in V_i are:

$$\begin{aligned}\mathbb{S}^{[l,m]}(x_1^{\pm1},\ldots,x_n^{\pm1}) &= \langle l|_i\,\mathbb{S}^{[i]}(x_1^{\pm1},\ldots,x_n^{\pm1})\,|m\rangle_i\,,\\ \mathbb{B}^{[0,0]}(x_1^{\pm1},\ldots,x_n^{\pm1};\gamma) &= \langle 0|_i\,\mathbb{B}^{[i]}(x_1^{\pm1},\ldots,x_n^{\pm1};\gamma)\,|0\rangle_i\,.\end{aligned}$$

Then we can re-express the H functions in terms of column transfer matrices instead of row transfer matrices:

$$\begin{aligned}&\prod_{i=1}^{n}(x_i-t\bar{x}_i)H_\lambda(x_1^{\pm1},\ldots,x_n^{\pm1};t;\gamma,\delta) =\\ &\qquad\langle\mathbb{K}|\,\mathbb{B}^{[0,0]}(x_1^{\pm1},\ldots,x_n^{\pm1};\gamma)\mathbb{B}^{[0,0]}(x_1^{\pm1},\ldots,x_n^{\pm1};\delta)\prod_{i=0}^{\infty}\mathbb{S}^{[m_i(\lambda),0]}(x_1^{\pm1},\ldots,x_n^{\pm1})\,|\bullet,\ldots,\bullet\rangle\,,\end{aligned}$$

where

$$\langle\mathbb{K}| = \bigotimes_{k=1}^{n}\langle K|_{k\bar{k}}\,. \tag{10}$$

We now define twisted versions of these operators using the double-row D matrix $\mathbb{D}_{1\ldots n}$:

$$\begin{aligned}\widetilde{\mathbb{S}}^{[i]}(x_1^{\pm1},\ldots,x_n^{\pm1}) &= \mathbb{D}_{1\ldots n}\mathbb{S}^{[i]}(x_1^{\pm1},\ldots,x_n^{\pm1})\mathbb{D}_{1\ldots n}^{-1},\\ \widetilde{\mathbb{B}}^{[i]}(x_1^{\pm1},\ldots,x_n^{\pm1};\gamma) &= \mathbb{D}_{1\ldots n}\mathbb{B}^{[i]}(x_1^{\pm1},\ldots,x_n^{\pm1};\gamma)\mathbb{D}_{1\ldots n}^{-1},\\ \langle\widetilde{\mathbb{K}}| &= \langle\mathbb{K}|\,\mathbb{D}_{1\ldots n}^{-1}.\end{aligned}$$

Now, as a consequence of (57) and (58) of [2], as well as (2), that:

Remark 1. *We have, where $x_{\bar{k}} = \bar{x}_k$,*

$$\begin{aligned}\widetilde{\mathbb{S}}^{[m,0]}(x_1^{\pm1},\ldots,x_n^{\pm1}) &= \sum_{\substack{I\subset\{1,\bar{1},\ldots,n,\bar{n}\}\\|I|=m}}\bigotimes_{i\in I}\begin{pmatrix}0 & \frac{x_i}{1-sx_i}\\ 0 & 0\end{pmatrix}_i\bigotimes_{j\notin I}\begin{pmatrix}\frac{x_j-s}{1-sx_j}\prod_{k\in I}\frac{x_j-tx_k}{x_j-x_k} & 0\\ 0 & 1\end{pmatrix}_j,\\ \widetilde{\mathbb{B}}^{[0,0]}(x_1^{\pm1},\ldots,x_n^{\pm1};\gamma) &= \bigotimes_{i\in\{1,\bar{1},\ldots,n,\bar{n}\}}\begin{pmatrix}1 & 0\\ 0 & 1-\gamma x_i\end{pmatrix}_i,\\ \prod_{i=1}^{n}\frac{1}{(x_i-t\bar{x}_i)}\langle\widetilde{\mathbb{K}}| &= \sum_{\{\epsilon_1,\ldots,\epsilon_n\}\in\{\pm1\}^n}\prod_{k=1}^{n}\left(\frac{x_k^{\epsilon_k}}{1-x_k^{2\epsilon_k}}\right)\prod_{1\leq k<l\leq n}\left(\frac{1-tx_k^{\epsilon_k}x_l^{\epsilon_l}}{1-x_k^{\epsilon_k}x_l^{\epsilon_l}}\right)\bigotimes_{k=1}^{n}\langle-\epsilon_k|_k\otimes\langle\epsilon_k|_{\bar{k}}.\end{aligned}$$

We can then similarly use (2) and Theorem 3 of [2] to obtain a symmetrization expression for H functions:

Theorem 3. *The H functions are given by:*

$$H_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; t, s; \gamma, \delta) =$$
$$\frac{(s^2;t)_\lambda}{v_\lambda(t)} \sum_{\omega \in H_n} \omega \left(\prod_{1 \le i \le n} \left(\left(\frac{x_i - s}{1 - s x_i} \right)^{\lambda_i} \frac{(1-\gamma\bar{x}_i)(1-\delta\bar{x}_i)}{(1-\bar{x}_i^2)(1-s x_i)} \right) \prod_{1 \le i < j \le n} \frac{(x_i - t x_j)(1 - t\bar{x}_i\bar{x}_j)}{(x_i - x_j)(1-\bar{x}_i\bar{x}_j)} \right).$$

Proof. The proof is analogous to Lemma 4 and Remark 5 of [2]. □

4 Discussion and Future Work

Future work may include creating dual functions to the F and H functions, analogous to the Q and L functions in [2], using L^* matrices instead of L matrices. These dual functions could then be used to prove analogues of the identities in [2]. Also, skew F and H functions like those in [3] can be created by allowing both end conditions, rather than just one, in the graphical representations to vary. It then may be possible to use this to extend some identities in [3] from type A to type BC polynomials.

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