Coefficients of q-binomial coefficients modulo N

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What is a q-analogue?

A q-*analogue* is an extension of a combinatorial object based on a variable q reducing to the initial object at $q = 1$.

Example

We can generalize n as a polynomial $[n]_q$ given by

$$(\mathbf{n})_q = \frac{1 - q^n}{1 - q} = 1 + q + \ldots + q^{n-1}.$$

Note that at $q = 1$ we obtain n.

Defining a q-binomial coefficient

Note that

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

This motivates defining $[\mathbf{n}]!_q = [n]_q[n-1]_q \cdots [1]_q$, and letting

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[n-k]!_q[k]!_q}.$$

We can then say:

- $\binom{n}{k}$ as the $q = 1$ case
- $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number of k-dimensional subspaces of $\mathbb{F}_q^n$.

Defining $f_k(n)$

The original problem concerns looking at the coefficients of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ modulo two.

Definition

For $R \in \mathbb{Z}/2\mathbb{Z}$, and $k \in \mathbb{N}$, we define

$$f_k(n) = \#\left\{\alpha : [q^\alpha] \begin{bmatrix} n \\ k \end{bmatrix}_q \equiv R \pmod{2}\right\}.$$

The function counts the number of coefficients congruent to R modulo 2.

Example

Let's compute $f_2(5)$. We have

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$$

Motivation

- Nice patterns in binomial coefficients $\binom{n}{k}$:

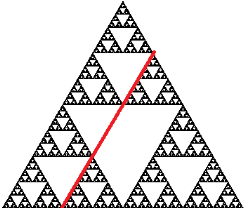

Figure 1: Pascal's triangle modulo two

- $\binom{n}{k}$ is the sum of the coefficients of $\begin{bmatrix} n \\ k \end{bmatrix}_q$, or $\sum_{i\geq 0}[q^i]\begin{bmatrix} n \\ k \end{bmatrix}_q$, so we should expect some sort of pattern with coefficients of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ as well.

We show that $f_k(n)$ is

- *quasipolynomial* - we can say

$$f_k(n) = P_i(n) \text{ when } n \equiv i \pmod{Q}$$

  where $P_i \in \mathbb{Z}[x], Q \in \mathbb{N}$.

- We call Q a *quasiperiod*, and the degree is $\max_i (\deg P_i)$.
- Example: $\lfloor n/2 \rfloor$ is quasipolynomial of degree 1 and quasiperiod 2.

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Theorem

The function $f_k(n)$ counting coefficients congruent to some fixed residue modulo N in $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is quasipolynomial of degree one. That is, it can be written as for

$$f_k(n) = \ell_i(n)$$

for $n \equiv i \pmod{Q_k}$ and linear functions $\ell_i(n)$.

What do the slopes look like?

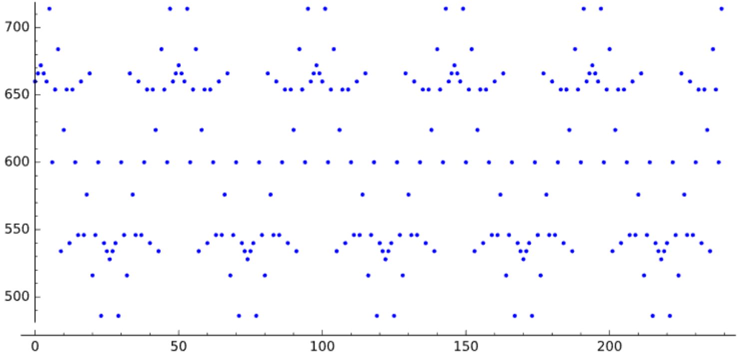

Figure 2: Slopes of f_5 (counting odd coefficients of $\begin{bmatrix} n \\ 5 \end{bmatrix}_q$)

A combinatorial interpretation of $\begin{bmatrix} n \\ k \end{bmatrix}_q$

Definition

A *partition* λ of n is a way to write n as a sum of positive integers $\lambda_1, \lambda_2 \ldots$ where **order does not matter**.
A *part* of λ is one of these integers λ_i.

A **Young diagram** is a way of representing a partition λ. For example, take the partition $1 + 2 + 4$ of 7:

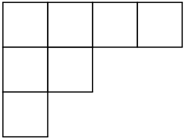

Figure 3: A partition of 7

A combinatorial interpretation of $\begin{bmatrix} n \\ k \end{bmatrix}_q$

Theorem

We have

$$\sum_{n\geq 0} p_{j\times k}(n)q^n = \begin{bmatrix} j+k \\ k \end{bmatrix}_q,$$

where $p_{j\times k}(n)$ is the number of partitions whose Young diagrams fit in a $j \times k$ box.

We can use this to show that the coefficients of $\begin{bmatrix} j+k \\ k \end{bmatrix}_q$ are symmetric.

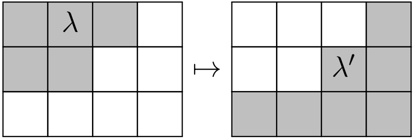

Figure 4: The map $\lambda \mapsto \lambda'$ on a 4x3 box

Coefficients of low degree terms in $\begin{bmatrix} n \\ k \end{bmatrix}_q$

Fix $k = 3$, and make a list of the first 10 terms of $\begin{bmatrix} n \\ 3 \end{bmatrix}_q$:

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix}_q = \ldots + 0q^9 + 0q^8 + 0q^7 + q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$$

$$\begin{bmatrix} 10 \\ 3 \end{bmatrix}_q = \ldots + 10q^9 + 9q^8 + 8q^7 + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + q + 1$$

$$\begin{bmatrix} 15 \\ 3 \end{bmatrix}_q = \ldots + 12q^9 + 10q^8 + 8q^7 + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + q + 1$$

$$\begin{bmatrix} 20 \\ 3 \end{bmatrix}_q = \ldots + 12q^9 + 10q^8 + 8q^7 + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + q + 1$$

Periodicity of $p_{\leq k}(n)$

Theorem (Nijenhuis & Wilf, 1987)

- *The function $p_{\leq k}(n)$ is **purely periodic** modulo N.*
- *This period is given by a function $\pi_N(k)$, which can be described explicitly.*

- This result is crucial in proving the main theorem, and the actual quasiperiod Q_k examined is loosely based on $\pi_N(k)$.
- Roughly speaking, we can can use $p_{\leq k}(n)$ as a "basis" and exploit its periodicity modulo N to prove the main theorem.

What does the repeating sequence look like modulo N?

Theorem

Let $q_0, q_1, \ldots q_{\pi_N(k)-1}$ be the repeating sequence of residues of $p_{\leq k}(n)$ modulo N, with $q_i \equiv p_{\leq k}(i) \pmod{N}$. Then:

- *The last $\binom{k+1}{2} - 1$ entries are zero.*
- *Ignoring the zero entries at the end, the residues exhibit symmetry.*

Example

One example of this sequence for $N = 2, k = 3$ is

q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}
1	1	0	1	0	1	1	0	0	0	0	0

The generating function

f_k being a degree one quasipolynomial allows for easy computation of the generating function:

Theorem

For a modulus $N \in \mathbb{N}$, we have

$$F_k(x) := \sum_{n \geq 0} f_k(n) x^n = \frac{A(x)}{(1 - x^{\pi'_N(k)})^2},$$

where

$$A(x) = \sum_{i \in \mathbb{Z}/Q_k\mathbb{Z}} (1 - x^{Q_k}) c_0(i) x^i + c_1(i) x^{Q_k + i}.$$

- $c_0(i)$ is the constant term for residue i.
- $c_1(i)$ is the slope for residue i.

Future directions

- Prove that the numerator of $F_k(x)$ is usually symmetric. For example, when $N=2$ when we count the residue 1 we get

$$F_3(x) = \frac{x^{15} + 4x^{14} + 4x^{13} + \cdots + 4x^5 + 4x^4 + x^3}{\Phi_1(x)^2\Phi_2(x)^2\Phi_3(x)\Phi_4(x)^2\Phi_6(x)\Phi_{12}(x)}$$

  Not always true, but true in many cases.
- Slopes appear to be bounded within a certain range. Is an asymptotic estimate possible?
- Is there a "nicer" formula for the generating function?

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References

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