Random walks on a Grid with a Periodic Boundary Condition

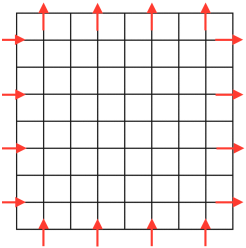
Peter Rowley

Mentor Boya Song

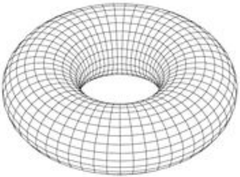
PRIMES Conference May 20, 2017

Periodic Boundary Condition

- Boundaries wrap to the other side
- Equivalent to a torus



(a) An $N \times N$ grid

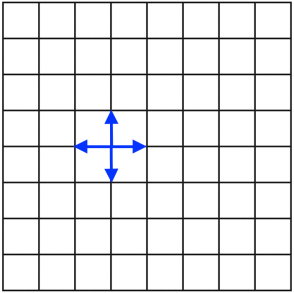


(b) A $\mathbb{Z}_N \times \mathbb{Z}_N$ torus

Figure: Two ways of viewing the grid

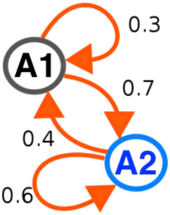
Random Walk

- Simple symmetric random walk: starting anywhere, at every step there is a 25% chance of moving in each direction (up, down, left, right)



Markov Chains

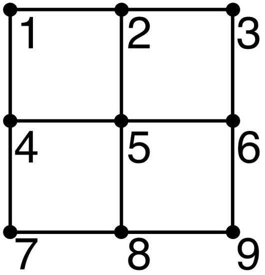
- A set of discrete states with probabilities to move between
- Irreducible if it is possible to get from any state to any other
- Can be modelled by a transition matrix
  - Columns add to 1
  - Element in ith row and jth column is probability of transition from jth state to ith state
  - Regular if some power has all positive entries



$$T = \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}$$

Figure: A Markov process and its transition matrix

3 by 3 Transition Matrix



$$T = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix}$$

Figure: Transition matrix for random walk on 3×3 grid

Steady State Distribution

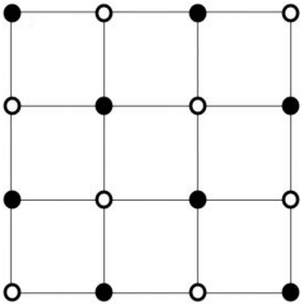
Definition

Steady State Distribution: A probability distribution of a Markov chain which stays constant when the transition matrix is applied

- Due to symmetry and reversibility of this random walk, the steady state distribution is all equal probabilities of $\frac{1}{n^2}$

The Even Case

- Grid can be colored black and white so that it always goes from black to white
  - Graph of states is bipartite
- Probability distribution does not approach a steady state vector
- We focus on the odd case

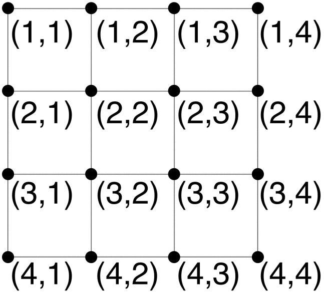


Eigenvalues

- It is known that all regular transition matrices have one eigenvalue of 1 and the rest satisfy $|\lambda| < 1$
- For small cases, we look at the number of distinct eigenvalues:
  - 3 by 3 has 3
  - 5 by 5 has 6
  - 7 by 7 has 10
  - 9 by 9 has 15
- We conjecture that for an odd $(2n+1) \times (2n+1)$ grid there are $\binom{n+2}{2}$ distinct eigenvalues

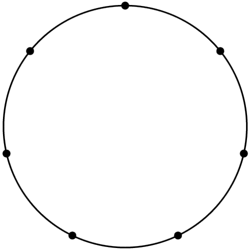
Viewing as a Product Chain

- Coordinates start with $(1,1)$ in top left, with (i,j) being ith row and jth column
- Can be seen as two separate random walks, one for each coordinate
- Each step randomly chooses one of the walks to increment
- Allows us to use results from random walk on a loop



Eigenvalues for Each Loop

- It is known that all distinct eigenvalues of a loop of length $2n+1$ are of the form $\cos\left(\frac{2\pi}{2n+1}k\right)$ for $0 \leq k \leq n$

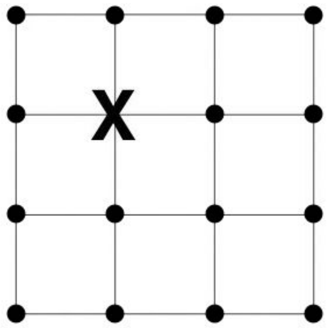


Combining the Eigenvalues

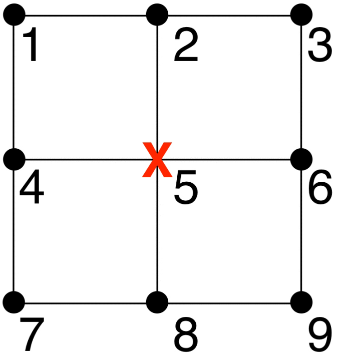
- In a product chain of d chains, if P is a probability distribution over the set of chains, and λ_i is any eigenvalue of the ith process, then
$$\sum_{i=1}^{d} P_i \lambda_i$$
is an eigenvalue of the product chain.
- Any λ_i, λ_j from have $\frac{\lambda_i + \lambda_j}{2}$ as an eigenvalue of the 2-D walk
- This gives $\binom{n+2}{2}$ distinct values

Removing a Point

- One point is removed
  - Impossible to move to or from that point



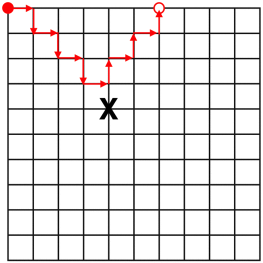
Transition Matrix



$$T = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{3} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{3} & 0 \end{bmatrix}$$

Time to Affect

- Probability not affected for states not near the removed point at first
- Comparing the probabilities of being at any given point after a certain amount of time
- Only affected once the path can have traveled to a point adjacent to the removed point



Future Research

- Consider eigenvalues of the even case
- Consider eigenvalues of the point-removed case
- Look into the expected hitting times with and without a point removed

Acknowledgements

- My mentor Boya Song
- Professor Jörn Dunkel
- MIT Math Department
- MIT PRIMES
- Dr. Tanya Khovanova
- Dr. Slava Gerovitch
- My parents