Elliptic Curves and Mordell's Theorem

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MIT PRIMES

December 16, 2017

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Definition (Diophantine Equations)

Diophantine Equations are polynomials of two or more variables with solutions restricted to $\mathbb{Z}$ or $\mathbb{Q}$.

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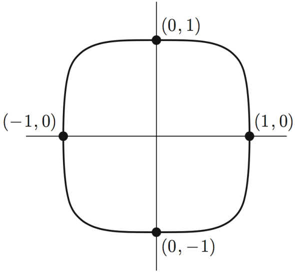
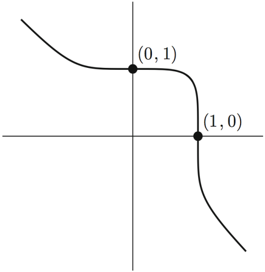
- Ex. Fermat's theorem: $x^n + y^n = 1,\ n > 2,\ x, y \in \mathbb{Q}$ equivalent to $x^n + y^n = z^n,\ x, y, z, \in \mathbb{Z}$

The Rational Points on Fermat Curves

Two examples of Diophantine equations with rational solutions marked: $x^4 + y^4 = 1$ and $x^5 + y^5 = 1$.





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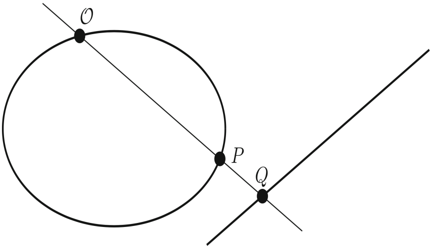
- Question: finite or infinite number of rational points?
- Question: given some known rational points on a curve, can we generate more?
- Mordell's Theorem: finite number of rational points generate all rational points for a class of cubic curves (elliptic curves)

Rational Points on Conics

Definition

General Rational Conic: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, $A, B, C, D, E, F \in \mathbb{Q}$.



Theorem

Take a general conic with rational coefficients and a rational point $\mathcal{O}$. A point P on the conic is rational if and only if the line through P and $\mathcal{O}$ has rational slope.

- Theorem gives geometric method for generating rational points
- Method can be described algebraically

An Application: Generating Pythagorean Triples

Examples

Take the unit circle with $\mathbb{O} = (-1, 0)$. The line through $\mathbb{O}$ with rational slope t intersects the circle again at $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$.

Theorem (Generation of Pythagorean Triples)

(a, b, c) is an in integer solution to $x^2 + y^2 = z^2$ if and only if $(a, b, c) = (n^2 - m^2, 2mn, n^2 + m^2)$ for $n, m \in \mathbb{Z}$.

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- We see that this implies $c = n^2 + m^2$ and the rest follows

Rational Points on $y^2 = x^3 + c$

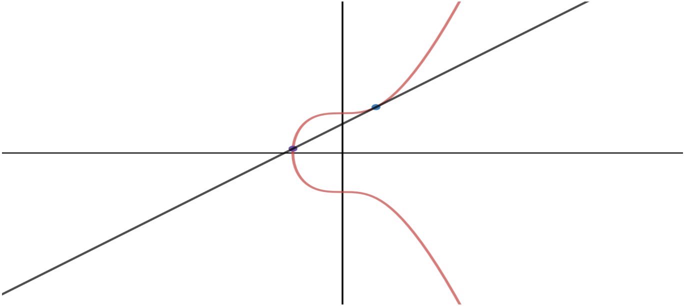
- Moving to cubics, our method for conics fails
- Given one rational point on a cubic curve, can we get more?
- Bachet studied rational solutions to $C : y^2 = x^3 + c$ for $c \in \mathbb{Z}$
- Discovered formula in (1621!) that takes one rational point on C and returns another

Bachet's Formula

Theorem (Bachet's Formula)

Bachet's formula says that for a cubic $C : y^2 = x^3 + c$ with $c \in \mathbb{Z}$, if (x_1, y_1) is a rational solution of C, then so is $\left(\frac{x^4-8cx}{4y^2}, \frac{-x^6-20cx^3+8c^2}{8y^3}\right)$.

There is a geometric procedure equivalent to applying Bachet: find the second intersection of the tangent at (x_1, y_1) and C.



Bachet's Formula

Take the example $C : y^2 = x^3 + 3$. One rational point by inspection is $(1, 2)$. Applying Bachet's formula yields

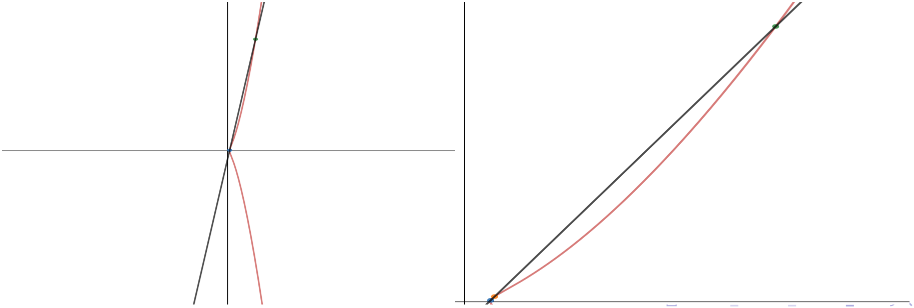
- $(1, 2)$
- $\left(-\frac{23}{16}, -\frac{11}{64}\right)$
- $\left(\frac{2540833}{7744}, -\frac{4050085583}{681472}\right)$
- And so on... This formula almost always generates infinitely many rational points.

Can often find one solution by inspection, so being able to generate infinitely many is a huge improvement.
But Bachet does not generate all solutions.

But! Bachet is Not Enough

$y^2 = x^3 - 26$ has two "easy" rational roots: $(3,1)$ and $(35,207)$. Applying Bachet to each repeatedly:

- $(3,1) \to \left(\frac{705}{4}, \frac{18719}{8}\right) \to \left(\frac{247043235585}{5606415376}, \frac{-122770338185379457}{419785957693376}\right) \to \ldots$



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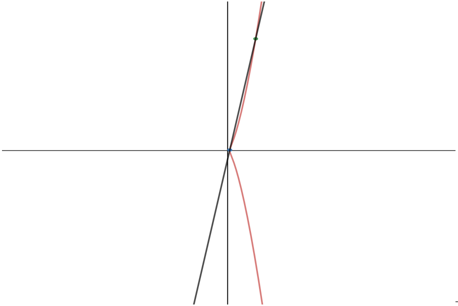

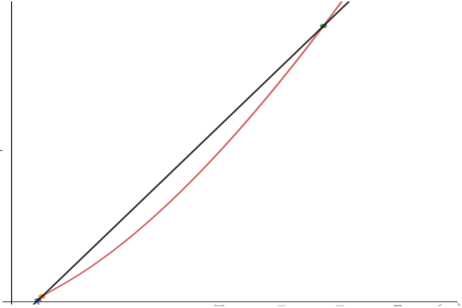

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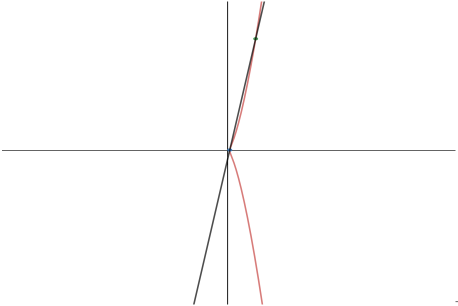
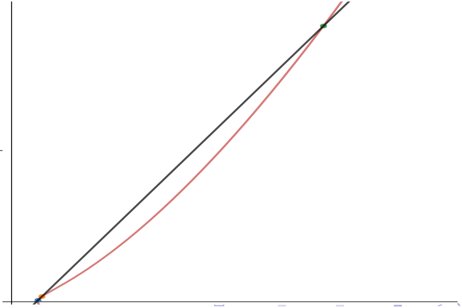
- The line through $(3,1)$ (blue) and $(35,207)$ (green) intersects C at $\left(\frac{881}{256}, \frac{15735}{4096}\right)$ (orange).





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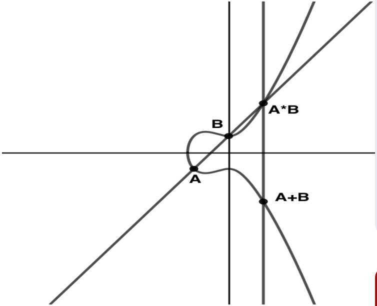
- $\left(\frac{881}{256}, \frac{15735}{4096}\right)$ does not show up in either sequence
- But it **can** be generated from $(3,1)$ and $(35,207)$
- We need a method for generating new rational points from 2 inputs

Group Law



Definition (The Group Law on Rational Points in C)

Let distinct $A, B \in C$ have coordinates in $\mathbb{Q}$. Define $A + B$ as the reflection over the $x - axis$ of the third intersection point, $A * B$, of line $\overline{AB}$ with C. If $A = B$, we define $A + B$ as the reflection of the second intersection point of the tangent line to C at A with C.

The Identity

We define the identity as $\mathcal{O}$. If A and B share a x-coordinate, we say $\overline{AB}$ intersects C "at infinity" at $\mathcal{O}$.

Rational Elliptic Curves

We can generalize Bachet's formula to more general cubics, namely rational elliptic curves.

Definition (Rational Elliptic Curves)

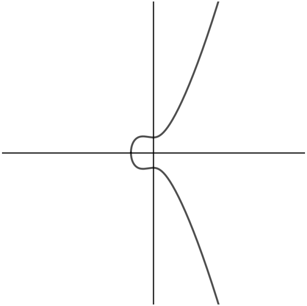
We define rational elliptic curves as non-singular algebraic plane curves described by polynomials of the form $y^2 = x^3 + ax^2 + bx + c,\ a, b, c \in \mathbb{Q}$, plus a "point at infinity" $\mathcal{O}$.

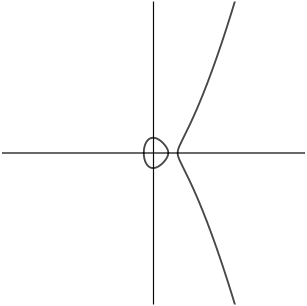
Definition

The group of rational points on an elliptic curve C is denoted by $C(\mathbb{Q})$.

Examples

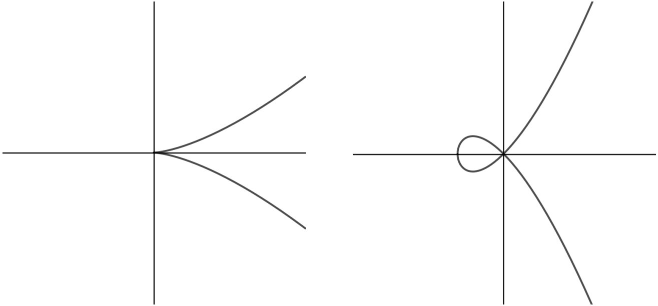
Below are the graphs of two elliptic curves in $\mathbb{R}^2$: $y^2 = x^3 + x^2 + 1$ and $y^2 = x^3 - 2x^2 + 1$.





Non-Examples

These curves are singular and therefore are **not** elliptic curves: $y^2 = x^3$ and $y^2 = x^3 + x^2$. Notice that all have either a cusp, or self-intersection (node).



Finite Generation

We are interested in the *generation* of $C(\mathbb{Q})$.

Definition

A group G is finitely generated if there exists a finite set $\{g_1, g_2, ..., g_n\} \subset G$ such that for all $a \in G$ there exist $\{a_1 ... a_n\} \subset \mathbb{Z}$ such that $a = \sum_{i=0}^{n} g_i a_i$.

Mordell's Theorem

Theorem (Mordell's Theorem)

Let C be a non-singular cubic curve given by an equation

$$C : y^2 = x^3 + ax^2 + bx,$$

with $a, b \in \mathbb{Z}$. Then $C(\mathbb{Q})$, the group of rational points on C, is a finitely generated abelian group.

- Restricted to elliptic curves with a root at $(0, 0)$.
- This means there exists a finite set of points so that all rational points can be obtained by inductively applying the group law.

Proof of Mordell's Theorem:

- Consider the subgroup $2C(\mathbb{Q})$ of $C(\mathbb{Q})$. Then take representatives $A_1, A_2, \ldots$ of its cosets.

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- Repeating m times and back-substituting,

$$P = A_{i_1} + 2A_{i_2} + 4A_{i_3} + \ldots + 2^{m-1}A_{i_m} + 2^m P_m$$

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$\exists$ *finite* S *independent of* P *such that for large enough* m, $P_m \in S$.

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- $$\left(\frac{129}{100}, \frac{-383}{1000} \right) = 2(3,5) + 0$$

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- Notice numerators and denominators decrease as m increases
- $\exists K \in \mathbb{Z}$ dependent only on C such that for sufficiently large m, numerator and denominator of x-coordinate of P_m less than K
- S is the set of $P \in C(\mathbb{Q})$ with x-coordinate's with numerator and denominator less than K

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- Lemma 1 tells us there is a finite set S of P_m.

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The number of cosets of $2C(\mathbb{Q})$ in $C(\mathbb{Q})$ is finite.

- Equivalent to the index $(C(\mathbb{Q}) : 2C(\mathbb{Q}))$ being finite.
- This result is known as Weak Mordell's Theorem

Note that

$$P = A_{i_1} + 2A_{i_2} + 4A_{i_3} + ... + 2^{m-1}A_{i_m} + 2^m P_m.$$

- Lemma 1 tells us there is a finite set S of P_m.
- Lemma 2 tells us that there is a finite set of A_i.

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- Lemma 1 tells us there is a finite set S of P_m.
- Lemma 2 tells us that there is a finite set of A_i.
- Thus, generating set $G = S \cup \{A_1, A_2, \ldots\}$ is finite.

Generalizations

- Mordell's theorem holds for all rational elliptic curves, not only those with a root at $(0, 0)$.
- Mordell made a conjecture about higher degree curves that was proved in 1983 by Falting.

Theorem

Falting's Theorem] A curve of genus greater than 1 *has only finitely many rational points.*

Definition (Genus)

The genus g of a non-singular curve can be defined in terms of its degree d as $\frac{(d-1)(d-2)}{2}$.

Notice that elliptic curves therefore have genus 1.

Acknowledgements

We would like to thank

- Our mentor, Andrew Senger

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- Our mentor, Andrew Senger
- MIT PRIMES
- Our parents