PRIMES 2017 Math Problem Set

Dear PRIMES applicant:

This is the PRIMES 2017 Math Problem Set. Please send us your solutions as part of your PRIMES-USA application by the application deadline (November 15, 2016). For complete rules, see `http://math.mit.edu/research/highschool/primes/usa/apply-usa.php`

Note that this set contains two parts: "General Math problems" and "Advanced Math." Please solve as many problems as you can in both parts.

You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF file. The name of the attached file must start with your last name, for example, "`smith-solutions`". Include your full name in the heading of the file.

Please write not only answers, but also proofs (and partial solutions / results / ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.

You are allowed to use any resources to solve these problems, *except other people's help.* This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

Note that posting these problems on problem-solving websites before the application deadline is strictly forbidden! Applicants who do so will be disqualified, and their parents and recommenders will be notified.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps several days. We encourage you to apply if you can solve at least 60% of the problems.

We note, however, that there will be many factors in the admission decision besides your solutions of these problems.

Enjoy!

General Math Problems

Problem G1. A positive multiple of 11 is *good* if it does not contain any even digits in its decimal representation.

(a) Find the number of good integers less than 1000.

(b) Determine the largest such good integer.

(c) Fix $b \geq 2$ an even integer. Find the number of positive integers less than b^3 which are divisible by $b+1$ and do not contain any even digits in their base b representation. (This is the natural generalization of part (a) with 10 replaced by b.)

Problem G2. A fair six-sided die whose sides are labelled 1, 4, 9, 16, 25, 36 is rolled repeatedly until the sum of the rolled numbers is nonzero and either even or a multiple of 3.

(a) Compute the probability that when we stop, the sum is odd.

(b) Find the expected value of the number of rolls it takes until stopping.

Partial marks may be awarded for approximate answers obtained by computer simulation. (This is also a good way to check your answer!)

Problem G3. Let $\mathcal{H}$ be a hyperbola with center Z. Points A and B are selected on $\mathcal{H}$. Suppose that the tangents to $\mathcal{H}$ at points A and B intersect at a point C distinct from A, B, Z. Prove that line ZC passes through a point X in the interior of segment AB and determine the ratio AX/AB.

Problem G4. Suppose $P(n)$ is a monic polynomial with integer coefficients for which $P(0) = 17$, and suppose distinct integers $a_1, \ldots, a_k$ satisfy $P(a_1) = \cdots = P(a_k) = 20$.

(a) Find the maximum possible value of k (over all P).

(b) Determine all P for which this maximum is achieved.

Problem G5. A positive integer N is *nice* if all its decimal digits are 4 or 7.

(a) Find all nine-digit nice numbers which are divisible by 512.

(b) How many d-digit nice numbers are divisible by 512 for each d?

Problem G6. A sequence x_1, x_2, $\ldots$ is defined by $x_1 = 10$ and

$$x_n = 3n \cdot x_{n-1} + n! - 3^n \cdot (n^2 - 2)$$

for integers $n \geq 2$. Derive a closed form for x_n (not involving Σ summation).

Problem G7. Let ABC be a triangle, let a, b, c be the lengths of its sides opposite to A, B, C respectively, and let h_A, h_B, and h_C be the lengths of the altitudes from A, B, and C. Suppose that

$$\sqrt{a + h_B} + \sqrt{b + h_C} + \sqrt{c + h_A} = \sqrt{a + h_C} + \sqrt{b + h_A} + \sqrt{c + h_B}. \quad (1)$$

(a) Show that $(a + h_B)(b + h_C)(c + h_A) = (a + h_C)(b + h_A)(c + h_B)$.

(b) Prove that the three terms on the left-hand side of (1) are obtained by a permutation of the three terms on the right-hand side of (1). (Possible hint: consider polynomials with the three terms as roots.)

(c) Show that triangle ABC is isosceles.

Advanced Math Problems

Problem M1. For every positive integer n set

$$a_n = 1^{-2} + 2^{-2} + \cdots + n^{-2}.$$

(a) Prove that the infinite sum

$$\sum_{n \geq 2} \frac{1}{n^2 a_n a_{n-1}}$$

is convergent.

(b) Determine its value.

Problem M2. Let $n \geq 1$ be a positive integer. An $n \times n$ matrix M is generated as follows: for each $1 \leq i, j \leq n$, we randomly and independently write either i or j in the (i,j)th entry, each with probability $\frac{1}{2}$. Let E_n be the expected value of $\det M$.

(a) Compute E_2.

(b) For which values of n do we have $E_n \geq 0$?

Problem M3. Let $k \geq 1$ be a positive integer. Find an $\varepsilon > 0$ in terms of k, as large as you can, such that the following statement is true.

> Consider a family $\mathscr{F}$ of subsets of $S = \{1, 2, \ldots, N\}$, where $N > k$ is an integer. If for any $T \subseteq S$ with $1 \leq |T| \leq k$ we have
>
> $$\frac{1}{2} - \varepsilon < \frac{|\{X \in \mathscr{F} : |X \cap T| \text{ is odd}\}|}{|\mathscr{F}|} < \frac{1}{2} + \varepsilon,$$
>
> then some set in $\mathscr{F}$ contains none of $\{1, \ldots, k\}$.

Problem M4. There are $n \geq 3$ married couples attending a daily couples therapy group. Each attendee is assigned to one of two round tables, so that no one sits at the same table with his/her spouse. The order of seating at each table remains fixed once and for all.

Initially, s of the attendees have contracted a contagious disease. For each person P, consider P's two neighbors at the table, as well as P's spouse (who, of course, sits at the other table). Each day, if at least two of these three people are sick, then P gets sick too, and remains sick forever.

Eventually everyone gets sick. Across all possible seating arrangements, what is the smallest possible value of s?

Problem M5. Consider an $n \times n$ binary matrix T (all entries are either 0 or 1). Assume at most $0.01n^2$ of the entries of T are zero.

(a) Find a constant $c > 0$, as large as you can, such that: for integers $m \geq 2$, the trace of T^m is at least $(cn)^m$.

(b) Does any $c < 1$ work?

Problem M6. You're given a simple graph G with n vertices, and want to develop an algorithm which either finds a 4-cycle or proves that none exists. For example, a naïve algorithm taking $O(n^4)$ runtime would be to simply brute-force search all $\binom{n}{4}$ possible sets of four vertices.

(a) Exhibit an algorithm with the best runtime you can find.

(b) Give the best lower bounds you can on the runtime of any such algorithm.

Open-ended problem. Research in mathematics is different from problem solving. One of the main differences is that you can ask your own questions. This problem is unusual. You do not need to solve anything, you have to invent questions and show your vision.

We start with a famous problem:

> You have a segment $[0, 1]$. You choose two points on this segment at random. They divide the segment into three smaller segments. What is the probability that the three smaller segments can be the sides of a triangle?

Imagine that you are a mentor and want to create a research project related to this problem. Your task is to invent up to five questions based on this problem.

Let us give you an example:

Question. The problem is equivalent to asking the probability that none of the three smaller segments exceed $1/2$ in length. A new question might be: Given x, what is the probability that none of the three smaller segments exceed x in length?

This question is a rather trivial generalization, which has an easy solution. Your goal is to invent something more challenging and stimulating. This problem will be judged separately based on how interesting your questions are.

To test how good your question is, it's a good idea to think about it a little bit, consider the simplest examples, etc. If this leads you to solving the question completely, then the question was too easy, and you should think how to make it harder! Please discuss the insight gained this way for each question.