Bounds on the Maximal Cardinality of an Acute Set in a Hypercube

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Abstract

The acute set problem asks the following question: what is the maximal cardinality of a d-dimensional set of points such that all angles formed between any three points are acute? In this paper, we consider an analogous problem with the condition that the acute set is a subset of a d-dimensional unit hypercube. We provide an explicit construction and proof to show that a lower bound for the maximum cardinality of an acute set in $\{0,1\}^d$ is $2^{2^{\lfloor \log_3 d \rfloor}}$. Using a similar construction, this lower bound is improved to $2^{d/3}$. Through a consideration of points diagonally opposite a particular point on 2-faces, we improve the upper bound to $\left(1+\dfrac{2}{d}\right)\cdot 2^{d-2}$. We then seek to generalize these findings and a combinatorial interpretation of the problem in $\{0,1\}^d$.

1 Introduction

In 1972, Danzer and Grünbaum [1] proposed the following question: what is the maximal cardinality $f(d)$ of a set of points in d dimensions such that any three points form an acute triangle? Danzer and Grünbaum proved that $f(d) \geq 2d-1$ and conjectured that this bound was the sharpest lower bound. In 1983, Erdős and Füredi [2] disproved this conjecture and provided an exponential bound for higher dimensions: $f(d) \geq \dfrac{1}{2}\left(\dfrac{2}{\sqrt{3}}\right)^d$. In 2011, Harangi [3] improved the bound to $f(d) \geq c\left(\sqrt[10]{\dfrac{144}{23}}\right)^d$. In April 2017, Zakharov [4] improved this bound to $f(d) \geq 2^{d/2}$. In September 2017, Gerencsér and Harangi [5] provided a construction with a $(d-1)$-dimensional hypercube to show that $f(d) \geq 2^{d-1}+1$. In particular, the growth rate of $f(d)$ is exactly 2 because the upper bound for the function $f(d)$ is 2^d, which is the cardinality of the set of points forming a d-hypercube.

As a result of the September 2017 finding, the problem of precisely bounding $f(d)$ may be reframed as: determine a function $1/2 \leq \alpha(d) \leq 1$ such that $f(d) \approx 2^d \cdot \alpha(d)$. Due to the difficulty of this problem, we considered the discretized acute set problem.

In this paper, we focus on a variant of the acute set problem — one in which the acute set is a subset of a $\{0,1\}^d$. First, we provide the preliminaries in Section 2, where we detail the discretized acute set problem and show a construction of an acute set with cardinality $d+1$ in $\{0,1\}^d$. We improve the result of Section 2 by providing in Section 3 a concrete example of a 9-dimensional acute set of size 16 on a unit hypercube. In Section 4, we present an explicit construction and proof to show that $2^{2^{\lfloor \log_3 d \rfloor}}$ is a lower bound. We then show that a similar construction results in a lower bound of $2^{d/3}$. Finally, in Section 5, we improve the upper bound of the maximal cardinality to $\left(1+\dfrac{2}{d}\right)\cdot 2^{d-2}$.

2 Maximal Cardinality of Acute Set on an Integer Lattice

Consider the integer lattice formed in d-dimensional space. As before, an acute set is a set of points such that all angles formed between any three points are acute. We now define the diameter of a d-dimensional acute set.

Definition 1. The *diameter* of an acute set is defined as $D(d,m) := \max(\sup|x_i - x_j|)$, where m is the cardinality of the acute set and x_i and x_j are points in the acute set.

In particular, we have that $D(d+1,m) \le D(d,m)$ and $D(d,m) \le D(d,m+1)$. In this paper, we consider the case in which the diameter of the acute set is 1, which is when the acute set is a subset of $\{0,1\}^d$. Through simple examples, one can observe that $D(d,d) = 1$ and, in particular, we have the following construction for $D(d,d+1)$. In studying the bounds of m such that $D(d,m) = 1$, we note that the lower bounds and proofs from previous results in $\mathbb{R}^d$ cannot serve immediate purpose in context due to the fact that much of those proofs rely on properties of $\mathbb{R}^d$, such as the slight perturbations of points on a hypercube.

Theorem 1. *In $\{0,1\}^d$, the set of points $\{(0,0,\ldots,0),\ (1,1,\ldots,1,0),\ \ldots,\ (0,1,\ldots,1)\}$, which consists of the origin and all points in the unit hypercube with exactly one 0 coordinate, forms an acute set for $d > 2$.*

Proof. Let x, y, and z be three elements of this set of points. We wish to show that for all such x, y, and z, $\langle y-x, z-x\rangle > 0, \langle x-y, z-y\rangle > 0$, and $\langle x-z, y-z\rangle > 0$. We now consider cases based on whether $(0,0,\ldots,0)$ is among the three-element subset chosen.

Case 1: $(\mathbf{0},\mathbf{0},\ldots,\mathbf{0})$ is one of the 3 points

If $(0,0,\ldots,0)$ is the middle point of an angle, then the vectors $\vec{v_1}$ and $\vec{v_2}$ originating from the origin have $d-1$ entries equal to 1 and the remaining entry as 0. In particular, since the position of the 0 entry is distinct between $\vec{v_1}$ and $\vec{v_2}$, $\langle\vec{v_1},\vec{v_2}\rangle = d-2$.

Now, we consider the case in which the point $(0,0,\ldots,0)$ is among the three points of consideration but is not a middle point. Then originating from some point will be one vector $\vec{v_1}$ composed of $d-1$ entries equal to -1 and one entry equal to 0 and another vector $\vec{v_2}$ composed of some ordering of $d-2$ entry equal to 0, one entry equal to 1, and the other entry equal to -1. The position of the 0 in $\vec{v_1}$ must correspond to the position of 1 in $\vec{v_2}$. The position of the -1 in the $\vec{v_2}$ corresponds to the position of -1 in $\vec{v_1}$. Hence, $\langle\vec{v_1},\vec{v_2}\rangle = 1$ and the angle formed between $\vec{v_1}$ and $\vec{v_2}$ is acute.

Case 2: $(\mathbf{0},\mathbf{0},\ldots,\mathbf{0})$ is not one of the 3 points

Suppose that none of the points for the angles in consideration are $(0,0,\ldots,0)$. Let x, y, and z be the three points. Without loss of generality, let y be the middle point. In the vectors $x-y$ and $z-y$, there are $d-2$ entries that are equal to 0, one entry equal to 1, and the remaining entry equal to -1. For all vectors formed by any set of three points, the arrangements of 0, 1, and -1 are distinct. In the vectors $x-y$ and $z-y$, the position of 1 in the ordering must be the same because in the same position of the 0 in point y. However, the position of -1 in the ordering will not be the same between $x-y$

and $z-y$. Hence, because each -1 is multiplied by 0 and each 1 is multiplied by 1, we have that $\langle x-y, z-y\rangle = 1$ and the angle $\angle xyz$ is acute.

$\square$

Let $h(d)$ be the maximal cardinality of an acute set in d-dimensional space such that $D(d, h(d)) = 2$. Through calculations found using a Python program, it was determined that for $d = 3$, $h(d) = 4$, for $d = 4$, $h(d) = 5$, and for $d = 5$, $h(d) = 6$. Although the data seem to suggest that $h(d)$ is a linear function (and, in particular, $d+1$), an explicit construction for an acute set in the 9-dimensional hypercube suggests that the size of the acute set grows at a rate much faster than linear.

3 Construction of an Acute Set in a 9-Dimensional Hypercube

The maximum cardinality of an acute set in a 3-dimensional cube is 4. Let $v_0 = (0,0,0)$, $v_1 = (1,1,0)$, $v_2 = (1,0,1)$, and $v_3 = (0,1,1)$ be the points in an acute set in the 3-dimensional cube. To construct an acute set with diameter 1 in 9-dimensional space, we consider the concatenation of points forming an acute set in the 3-dimensional cube. For instance, the point (v_0, v_0, v_0) represents the point $(0,0,0,0,0,0,0,0,0)$ in the 9-dimensional hypercube.

We note that the following set of concatenated points form an acute set in a 9-dimensional hypercube:

$$(v_0, v_0, v_0), (v_0, v_1, v_1), (v_0, v_2, v_2), (v_0, v_3, v_3)$$

$$(v_1, v_0, v_1), (v_1, v_1, v_2), (v_1, v_2, v_3), (v_1, v_3, v_0)$$

$$(v_2, v_0, v_2), (v_2, v_1, v_3), (v_2, v_2, v_0), (v_2, v_3, v_1)$$

$$(v_3, v_0, v_3), (v_3, v_1, v_0), (v_3, v_2, v_1), (v_3, v_3, v_2).$$

From this observation, it must be true that $h(9) \geq 16$. In Section 4, we generalize this construction to find a lower bound for the maximal cardinality of an acute set in $\{0,1\}^d$.

4 Lower Bound for $h(d)$

Let $S = \{v_0, v_1, \ldots, v_{h(d)-1}\}$ be the points in a set of maximal cardinality in a d-dimensional hypercube $\underbrace{\{0,1\} \times \cdots \times \{0,1\}}_{d \text{ dimensions}}$. Similar to the construction in Section 3, we consider the concatenation of points in the d-dimensional hypercube to form an acute set of points in a $3d$-dimensional hypercube $\underbrace{\{0,1\} \times \cdots \times \{0,1\}}_{3d \text{ dimensions}}$.

We now let $\phi(v)$ be a bijective function such that $\phi : \mathbb{Z}_{h(d)} \to S$ and $\phi(i) = v_r$ for some $v_i \in S$ and $r = i \pmod{h(d)}$. We concatenate 3 points in S to form the elements in the constructed set T, a subset of a $3d$-dimensional hypercube. We now claim that the following must be true of T.

Theorem 2. *Let T be the set constructed such that for all $v_i, v_j \in S$, $(v_i, v_j, \phi(i+j))$ is an element of T. Then, T is an acute set.*

Proof. Let w_i, w_j, and w_k be three points in the set T. We consider three different cases based on whether the points w_i, w_j and w_k share the same first d-dimensional point.

Case 1: w_i, w_j and w_k are all different for the first d dimensions

In this case, we have that the first d dimensions will always result in a positive inner product because the first d dimensions represent different elements of S. Thus, we must have that $\langle w_j - w_i, w_k - w_i\rangle > 0, \langle w_i - w_j, w_k - w_j\rangle > 0$, and $\langle w_i - w_k, w_j - w_k\rangle > 0$.

Case 2: w_i, w_j and w_k all have the same points for the first d dimensions

In this case, each of the second d dimensions will always be distinct because for all $t \in T$ there exists a unique pair of elements in S for the first two d-dimensional points of t. Thus, we must, again, have that $\langle w_j - w_i, w_k - w_i\rangle > 0, \langle w_i - w_j, w_k - w_j\rangle > 0$, and $\langle w_i - w_k, w_j - w_k\rangle > 0$.

Case 3: Two of w_i, w_j and w_k have the same points for the first d dimensions

In this case, there are three possibilities — one for each point being a middle point. Without loss of generality, let w_k be the point with different first d dimensions. Suppose that w_k is the middle point. Then, it must be true that $\langle w_i - w_k, w_j - w_k\rangle > 0$ because the first d dimensions result in a positive inner product. Now, without loss of generality, we consider the case in which w_j is the middle point (note that by symmetry, if the angle is acute when w_j is the middle point, the angle formed when w_i is the middle point will also be acute). Let $w_i = (v_a, v_b, \phi(a+b))$, $w_j = (v_a, v_c, \phi(a+c))$, and $w_k = (v_d, v_e, \phi(d+e))$. We note that $b \neq c$. If $e \neq b$ and $e \neq c$, we must have that $\langle w_i - w_j, w_k - w_j\rangle > 0$ because the second sets of d dimensions are distinct for w_i, w_j, and w_k. Otherwise, without loss of generality, let $e = c$. Then, $w_k = (v_d, v_c, \phi(d+c))$ and $\langle w_i - w_j, w_k - w_j\rangle > 0$ because the third set of d dimensions of w_j is distinct from that of w_i and w_k.

Therefore, T is an acute set. □

Remark. In this proof, we never rely on the fact that the acute set of points is a subset of a hypercube. In particular, we may recognize that such proof may hold in other metric spaces.

In Section 3, we constructed an acute set of 16 elements created by the function $\phi : \mathbb{Z}_4 \to S$, where $S = \{v_0, v_1, v_2, v_3\}$ and $\phi(i) = v_i$ for $i \in \mathbb{Z}_4$. In the case of $d = 3$, we had that $h(d) = 4$ and $h(3d) = 4 \cdot h(d) = 16$, the size of the acute set constructed. Here, we have generalized this construction for all d.

From this theorem, we have that if $h(d)$ is the maximal cardinality of an acute set in a d-dimensional hypercube, it must be true that $(h(d))^2 \leq h(3d)$ because the elements of T have every combination of indices in $\mathbb{Z}^2_{h(d)}$ for the first two points. As a result, we may conclude the following corollary.

Corollary 2.1. *If $h(n)$ is the maximal cardinality of an acute set in an n-dimensional hypercube, then $h(n) \geq 2^{2^{\lfloor \log_3 n \rfloor}}$.*

This result was first found in 2006 [6], in which the proof for Theorem 2 similarly employs a method of concatenating three points and and it was independently found in this research. However, using the same technique as before, we may improve the bound to $h(d) \geq 2^{d/3}$.

This time, we consider the concatenation of an element in S and two elements in an acute set in $\mathbb{R}^3$. Let $R = \{w_0, w_1, w_2, w_3\}$ be an acute set of maximal cardinality in $\{0,1\}^3$. Now, let $\psi(i)$ be a bijective function such that $\psi : \mathbb{Z}_4 \to R$ and $\psi(i) = w_i$. Now, let T be the set constructed such that for all $v_i \in S$, $w_j \in R$, $(v_i, w_j, \psi(i+j))$ is an element of T.

Through a proof provided in this paper and the paper published in 2006, we have the following theorem.

Theorem 3. *Let $d > 2$ and let $h(d)$ be the maximal cardinality of an acute set in a d-dimensional hypercube. Then, $h(d) \geq 2^{d/3}$.*

Observe that the bound in Theorem 3 is less than the lower bound for the maximal cardinality of an acute set in $\mathbb{R}^d$. Gerencsér and Harangi [5] constructed an acute set in $\mathbb{R}^{d-1}$ and perturbed points of a $(d-1)$-dimensional hypercube to satisfy the properties of an acute set. Here, in the discretized version of the acute set problem, such a perturbation is not possible.

5 Upper Bound for $h(d)$

To further understand the behavior of $h(d)$, we consider the upper bound of $h(d)$. With many restrictions in a d-dimensional hypercube, it is possible to decrease the upper bound for $h(d)$ from the known bound of 2^d.

In Theorem 4, we show that no two opposite points in $\{0, 1\}^d$ can both be elements of an acute set.

Theorem 4. *Let $d > 2$ be the dimension of a hypercube $\{0, 1\}^d$ and $h(d)$ be the maximal cardinality of an acute subset of the d-dimensional hypercube. Then, $h(d) \leq 2^{d-1}$.*

Proof. To decrease the upper bound, we consider the possibility of one point and its opposite point in the hypercube being in the acute set. Let $v = (a_0, a_1, \dots, a_{d-1})$ and $v' = (1 - a_0, 1 - a_1, \dots, 1 - a_{d-1})$ be two opposite points such that $a_i \in \{0, 1\}$. Now, let $w = (b_0, b_1, \dots, b_{d-1})$ be another point in the acute set such that $w \neq v$ and $w \neq v'$. We claim that $\langle v - w, v' - w\rangle = 0$.

For a given position, we have that if $a_i = 0$, then the corresponding position in $v - w$ would be $-b_i$ and the corresponding position in $v' - w$ would be $1 - b_i$. The corresponding term in the inner product would be 0.

Similarly, we have that if $a_i = 1$, then the corresponding position in $v - w$ would be $1 - b_i$ and the corresponding position in $v' - w$ would be $-b_i$. Again, the corresponding term in the inner product would be 0.

Hence, among any pair of opposite points in a d-dimensional hypercube, at most one point may be in the acute set and $h(d) \leq 2^{d-1}$.

□

From Theorem 3, we learn that $h(d)$ is strictly less than $f(d)$, the maximal cardinality of an acute set in $\mathbb{R}^d$, because $2^{d-1} + 1 \leq f(d)$ [5]. Furthermore, we find that no acute subset of a d-dimensional hypercube can have maximal cardinality in $\mathbb{R}^d$.

This upper bound can be further improved by fixing one point and considering the points diagonally opposite the point on 2-faces.

Theorem 5. *Let $d > 3$. Then, $h(d) \leq \left(1 + \dfrac{2}{d}\right) \cdot 2^{d-2}$.*

Proof. Without loss of generality, let the origin $v_0 = (0, 0, \dots, 0)$ be a point in the acute set. Then, we consider the points diagonally opposite to $(0, 0, \dots, 0)$ on all of the faces containing $(0, 0, \dots, 0)$. We further note that there are $\dbinom{d}{2}$ 2-faces of the hypercube that contain the origin. In particular, all such points diagonally opposite from the origin are exactly all points with exactly two positions equal to 1.

Without loss of generality, of the $\dbinom{d}{2}$ points diagonally opposite to $(0, 0, \dots, 0)$, we let $v = (1, 1, 0, \dots, 0)$ be a point in the acute set consisting of the first two coordinates equal to 1 and the rest equal to 0. We now consider another point w diagonally opposite to $(0, 0, \ldots, 0)$. In particular, we have $\langle v - v_0, w - v_0 \rangle > 0$, which means that there must be a common position in coordinates v and w equal to 1. Again, without loss of generality, let this new point $w = (1, 0, 1, 0, 0, \ldots, 0)$. Now, suppose that we add another new point. This new point must share a 1 in some common position with each of the other points. If the new point is of the form $(0, 1, 1, 0, \ldots, 0)$, then no other point can be an element of this acute set. Hence, all other points must be of the form $(1, 0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)$, which means that there are at most $d - 1$ points diagonally opposite from the origin on subfaces that are part of the acute set. Note that the choice of the fixed point in our acute set and the choice of which points diagonally opposite from that fixed point are elements of the acute set were arbitrary.

We now consider the maximum average number of points on a face that are elements of the acute set. From a given vertex, we have that there are $(d-1)$ 2-faces with 2 points and $\binom{d}{2} - (d-1)$ 2-faces with 1 point. Thus, the average number of points on a given face is:

$$\frac{(d-1) \cdot 2 + \binom{d}{2} - (d-1)}{\binom{d}{2}} = \frac{\binom{d}{2} + (d-1)}{\binom{d}{2}} = 1 + \frac{2}{d}.$$

The number of faces on a d-dimensional hypercube is $(d-1) \cdot d \cdot 2^{d-3}$. Taking the overcount into consideration, the total number of points is at most:

$$\frac{\left(1 + \frac{2}{d}\right) \cdot (d-1) \cdot d \cdot 2^{d-3}}{\binom{d}{2}} = \left(1 + \frac{2}{d}\right) \cdot 2^{d-2}.$$

In particular, if the total number of points is greater than $\left(1 + \frac{2}{d}\right) \cdot 2^{d-2}$, then the average number of points per face would be greater than $\left(1 + \frac{2}{d}\right)$, and this would violate the fact that there are at most $(d-1)$ points diagonally opposite from a point on a 2-face in the acute set by the Pigeonhole principle.

□

6 Conclusion

In this project, we studied the acute set problem in the integer lattice $\mathbb{Z}^d$, in which we introduced the diameter constraint to control the maximal cardinality of such an acute set. We then provided a construction of $d + 1$ points that would form an acute set in a d-dimensional hypercube. Although data in small dimensions seemed to suggest that the sharpest lower bound is $d+1$ (or $h(d)$ grows at a linear rate), we then provided an explicit construction of a 16-point acute set in a 9-dimensional hypercube from the points forming an acute set in a cube. From this construction, we generalized to conclude that it is possible to concatenate three points in a d-dimensional hypercube to form an acute set in a $3d$-dimensional hypercube. In particular, we improved the lower bound for the maximal cardinality of an acute set in a d-dimensional hypercube to $2^{2^{\lfloor \log_2 d \rfloor}}$ and, subsequently,

$2^{d/3}$. Regarding the upper bound of $h(d)$, we found that $h(d) \leq \left(1 + \frac{2}{d}\right) \cdot 2^{d-2}$ through a counting approach.

7 Future Work

For the future, perhaps one may consider the problem purely combinatorially. In particular, in the case of the acute set problem in the d-dimensional unit hypercube, the inner product of any two vectors in the hypercube is positive exactly when among any three points in the unit hypercube, there exist positions forming the set such that at least one is $\{1,1,0\}$ or $\{0,0,1\}$, at least one is $\{1,0,1\}$ or $\{0,1,0\}$, and at least one is $\{0,1,1\}$ or $\{1,0,0\}$. Such a question may perhaps be generalized with $k-1$ coordinates equal to 1 and 1 coordinate equal to 0 or $k-1$ coordinates equal to 0 and 1 coordinate equal to 1: is it true that the maximal cardinality of a set of points so that any $k-1$ dimensional hyperplanes also form acute angles is precisely the same as the maximal cardinality of a set of points so that there exist positions forming the set such that at least one is $\{1,1,\ldots,1,0\}$ or $\{0,0,\ldots,0,1\}$, at least one is $\{1,1,\ldots,1,0,1\}$ or $\{0,0,\ldots,0,0,1,0\}$, and so on? Perhaps solving this problem may lead to a combinatorial approach to precisely determining the maximal cardinality for acute sets.

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