Ramanujan Congruences for Fractional Partition Functions

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Overview

Definition of $p(n)$

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Example ($p(4) = 5$)

$$\begin{aligned} 4 &= 1+1+1+1 \\ &= 2+1+1 \\ &= 2+2 \\ &= 3+1 \\ &= 4. \end{aligned}$$

Generating function for $p(n)$

Lemma (Euler)

The generating function for $p(n)$ is

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

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We define

$$(q;q)_\infty := \prod_{n=1}^{\infty} (1-q^n).$$

Ramanujan's congruences

Theorem (Ramanujan (1915))

For every nonnegative integer n, we have

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

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Theorem (Watson (1938), Atkin (1967))

For prime $\ell \geq 5$ and positive integer r, define $0 \leq c_{\ell,r} < \ell^r$ such that $24c_{\ell,r} \equiv 1 \pmod{\ell^r}$. Then for every nonnegative integer n, we have

$$p(5^r n + c_{5,r}) \equiv 0 \pmod{5^r},$$
$$p(7^r n + c_{7,r}) \equiv 0 \pmod{7^{\lfloor r/2 \rfloor + 1}},$$
$$p(11^r n + c_{11,r}) \equiv 0 \pmod{11^r}.$$

A general theory of congruences

Theorem (Ahlgren, Ono (2000))

For every modulus L coprime to 6, there exist integers $A \neq 0$ and B such that for all n, we have

$$p(An + B) \equiv 0 \pmod{L}.$$

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Example

For all n, we have

$$p(4063467631n + 30064597) \equiv 0 \pmod{31}.$$

ℓ^r-balanced congruences

Definition

A congruence is ℓ^r-*balanced* if it is the form

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for all n, where c, r are integers and $r \geq 1$.

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Remark

The Ramanujan congruences and their generalizations to higher powers for $\ell = 5, 11$ are ℓ^r-balanced.

Questions inspired by the Ramanujan congruences

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Questions inspired by the Ramanujan congruences

1. Why do we have $24c \equiv 1 \pmod{\ell}$ for all congruences?
2. How many ℓ-balanced congruences are there for $p(n)$?
3. Is this a glimpse of a general theory of congruences?

Necessary condition for ℓ-balanced congruences

Question

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Theorem (Kiming-Olsson (1992))

Let $\ell > 5$ be a prime. If

$$p(\ell n + c) \equiv 0 \pmod{\ell}$$

for all n, then $24c \equiv 1 \pmod{\ell}$.

Finiteness for $p(n)$

Question

How many ℓ-balanced congruences are there for $p(n)$?

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Theorem (Ahlgren-Boylan (2001))

Let ℓ be prime. Then

$$p(\ell n + c) \equiv 0 \pmod{\ell}$$

for all n if and only if $(\ell, c) \in \{(5,4),(7,5),(11,6)\}$.

Fractional partition functions

Definition (Chan-Wang (2018))

The *fractional partition functions* $p_\alpha(n)$ are defined for rational $\alpha = a/b$ by

$$\sum_{n=0}^{\infty} p_\alpha(n) q^n := (q;q)_\infty^\alpha.$$

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Remark

- $\alpha = -1$ corresponds to usual partition function.
- $\alpha = -k \in \mathbb{Z}^-$ corresponds to k-colored partition function.

Denominators of $p_\alpha(n)$

Theorem (Chan-Wang 2018)

The denominator of $p_\alpha(n)$ when written in lowest terms is given by

$$\mathrm{denom}(p_\alpha(n)) = b^n \prod_{p \mid b} p^{\mathrm{ord}_p(n!)}.$$

Congruences for fractional partition functions

Theorem (Chan-Wang (2018))

For all n, we have

$$p_\alpha(\ell n + c) \equiv 0 \pmod{\ell},$$

if $24c \equiv -\alpha \pmod{\ell}$ and any of the following conditions hold:

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1. $\alpha \equiv 4, 8, 14 \pmod{\ell}$ and $\ell \equiv 5 \pmod{6}$;
2. $\alpha \equiv 6, 10 \pmod{\ell}$ and $\ell \equiv 3 \pmod{4}$ and $\ell \geq 5$;
3. $\alpha \equiv 26 \pmod{\ell}$ and $\ell \equiv 11 \pmod{12}$.

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Remark

Shortly, we will emphasize the special role of the list of α.

Examples from Chan-Wang

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1. $p_{-\frac{3}{4}}(43n + 39) \equiv 0 \pmod{43}$

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Remark

These congruences are ℓ-balanced.

Natural questions for rational α

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4. Is there an Ahlgren-Boylan analog (finiteness) for given α?

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Theorem 1 (BCC)

Let $\alpha = a/b$, and let $\ell \geq 5$ be a prime not dividing b such that $\alpha \not\equiv 1, 3 \pmod{\ell}$. If

$$p_\alpha(\ell n + c) \equiv 0 \pmod{\ell}$$

for all n, then $24c \equiv -\alpha \pmod{\ell}$.

Lacunary powers of the eta-function

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Lacunary powers of the eta-function

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Theorem (Serre 1985)

Let r be a positive even integer. Let

$$\eta := q^{1/24}(q;q)_\infty.$$

Then, η^r is lacunary if and only if

$$r \in \{2, 4, 6, 8, 10, 14, 26\}.$$

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Remark

The work of Chan and Wang relies on the identities that Serre proves to establish this theorem.

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Theorem 2 (BCC)

If ℓ is good for α with parameter k and $v \leq \mathrm{ord}_\ell(24k - \alpha)$ is a positive integer, then for all n, we have

$$p_\alpha(\ell^v n - k) \equiv 0 \pmod{\ell^v}.$$

Finiteness for integral α

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Theorem 3 (BCC)

Let α be an integer that is either even and < 0 or odd and > 3. If

$$p_\alpha(\ell n - \delta_\ell) \equiv 0 \pmod{\ell}$$

for all n, then $\ell \leq |\alpha| + 4$. In particular, p_α admits finitely many ℓ-balanced congruences.

Limiting residue classes of primes mod $2b$

Definition

For $m \in \mathbb{Z}^+$ and $\beta \in \mathbb{Q}$ with $\gcd(\mathrm{denom}(\beta), m) = 1$, define $\Psi_m(\beta)$:

- $\Psi_m(\beta) \in \{0, 1, \ldots, m-1\}$,
- $\Psi_m(\beta) \equiv \beta \pmod{m}$.

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- $\Psi_m(\beta) \in \{0, 1, \ldots, m-1\}$,
- $\Psi_m(\beta) \equiv \beta \pmod{m}$.

Theorem 4 (BCC)

Let $\alpha = a/b \in \mathbb{Q} - 2\mathbb{Z}$. If $\ell \geq |a| + 5b$ is a prime for which p_α admits an ℓ-balanced congruence, then

$$\Psi_{2b}\left(\frac{a}{\ell}\right) \geq b.$$

Modular forms and Hecke operators

Definition (Space of Modular Forms)

For $k \in 2\mathbb{Z}$, we let

- $M_k :=$ space of weight k modular forms on $\mathrm{SL}_2(\mathbb{Z})$,
- $S_k :=$ space of weight k cusp forms on $\mathrm{SL}_2(\mathbb{Z})$.

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Definition (Hecke Operators)

Let $f(z) = \sum_{n=0}^{\infty} a_n q^n \in M_k$, where $q := e^{2\pi i z}$. The *Hecke operator* T_ℓ acts via

$$(f \mid T_\ell)(z) = \sum_{n=0}^{\infty} \left(a(\ell n) + \ell^{k-1} a(n/\ell) \right) q^n.$$

Definition of Hecke eigenform

Definition

Let $f(z) = q + \sum_{n=2}^{\infty} a(n)q^n \in S_k$. We call $f(z)$ a *normalized Hecke eigenform* if for all m there exists $\lambda(m) \in \mathbb{C}$ such that

$$f(z) \mid T_m = \lambda(m) f(z).$$

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$$f(z) \mid T_m = \lambda(m) f(z).$$

Remark

There is a canonical basis of normalized Hecke eigenforms for S_k.

ℓ-non-ordinary primes

Definition

Let $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k \cap \mathcal{O}_L[[q]]$. We say that $f(z)$ is *ℓ-non-ordinary* if

$$a(\ell) \equiv 0 \pmod{\ell\mathcal{O}_L}.$$

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Remark

If $f(z)$ is a normalized Hecke eigenform, then

$$a(\ell n) = a(\ell)a(n) - \ell^{k-1}a(n/\ell).$$

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Remark

If $f(z)$ is a normalized Hecke eigenform, then

$$a(\ell n) = a(\ell)a(n) - \ell^{k-1}a(n/\ell).$$

Thus, ℓ-non-ordinarity is equivalent to

$$f(z) \mid T_\ell \equiv 0 \pmod{\ell\mathcal{O}_L}.$$

Extend powers of ℓ-non-ordinarity

Lemma (BCC)

Suppose $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k \cap \mathcal{O}_L[[q]]$ is an ℓ-non-ordinary normalized Hecke eigenform. Then for all $r, n \geq 1$,

$$a(\ell^r n) \equiv 0 \pmod{\ell^r \mathcal{O}_L}.$$

Hecke eigenforms ℓ-non-ordinary

Definition (ℓ good for α)

We say that a prime ℓ *is good for* $\alpha = a/b$ *with parameter* k if $\ell \nmid b$ and k is a positive integer such that

1. $\ell \mid (24k - \alpha)$,
2. $(\ell - 1) \mid (12k - m)$ for some $m \in \{4, 6, 8, 10, 14\}$, and
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Theorem (Jin, Ma, Ono 2016)

Let f be normalized Hecke eigenform of even weight $k \geq 12$. If $(\ell - 1) \mid (k - m)$ for some $m \in \{4, 6, 8, 10, 14\}$, then f is ℓ-non-ordinary.

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Remark

Let $f_1, \ldots, f_d$ be the basis of normalized Hecke eigenforms for S_k. For a cusp form $f(z) \in S_k \cap \mathcal{O}_L[[q]]$, we can write

$$f(z) = \sum_{i=1}^{d} \beta_i f_i.$$

By Cramer's rule, $\beta_i = \gamma_i / \mathcal{D}_k$ where $\gamma_i \in \mathcal{O}_L$.

ℓ-non-ordinarity extends to S_k

Lemma (BCC)

Let $k \geq 12$ be even and let ℓ be a prime such that

- $(\ell - 1) \mid (k - m)$ for some $m \in \{4, 6, 8, 10, 14\}$,
- $\ell \nmid N_k(\mathcal{D}_k)$.

Then for all $g(z) = \sum_{n=1}^{\infty} a_g(n) q^n \in S_k \cap \mathcal{O}_L[[q]]$, we have

$$a_g(\ell^r n) \equiv 0 \pmod{\ell^r \mathcal{O}_L}.$$

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$$a_g(\ell^r n) \equiv 0 \pmod{\ell^r \mathcal{O}_L}.$$

Remark

When $\ell \nmid N_k(\mathcal{D}_k)$ holds (condition 3 of ℓ being good for α), the ℓ-non-ordinarity of normalized eigenforms extends through linearity.

Proof of Theorem 2

Theorem 2 (BCC)

If ℓ is good for α with parameter k and $v \leq \mathrm{ord}_\ell(24k - \alpha)$ is a positive integer, then for all n, we have

$$p_\alpha(\ell^v n - k) \equiv 0 \pmod{\ell^v}.$$

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Ideas of Proof

- Technical lemma of Chan and Wang

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- Technical lemma of Chan and Wang
- Expression of ℓ^r-balanced congruences in terms of powers of Ramanujan's Delta-function $\Delta := q(q;q)_\infty^{24} \in S_{12}$

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Ideas of Proof

- Technical lemma of Chan and Wang
- Expression of ℓ^r-balanced congruences in terms of powers of Ramanujan's Delta-function $\Delta := q(q;q)_\infty^{24} \in S_{12}$
- ℓ-non-ordinarity of $\Delta^k \in S_{12k}$ implied by ℓ good for α with parameter k

Proof of Theorem 2

Lemma (Chan-Wang)

Let $\alpha = a/b$. Let ℓ be a prime not dividing b. Then for any $r \geq 1$,

$$(q;q)_\infty^{\ell^r \alpha} \equiv (q^\ell;q^\ell)_\infty^{\ell^{r-1}\alpha} \pmod{\ell^r}.$$

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Rewrite in terms of Ramanujan Δ-function

Write $r := \mathrm{ord}_\ell(24k - \alpha)$ and $\Delta^k =: \sum_{n=0}^\infty \tau_k(n) q^n$. Then,

$$\sum_{n=0}^\infty p_\alpha(n) q^{n+k} = q^k (q;q)_\infty^{24k+\ell^r u} \equiv \Delta^k (q^\ell;q^\ell)_\infty^{\ell^{r-1}u} \pmod{\ell^r}.$$

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Lemma (Chan-Wang)

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Rewrite in terms of Ramanujan Δ-function

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Extract terms of form $q^{\ell n}$ and replace q^ℓ by q:

$$\sum_{n=0}^{\infty} p_\alpha(\ell n - k) q^n \equiv (q;q)_\infty^{\ell^{r-1}u} \sum_{n=0}^{\infty} \tau_k(\ell n) q^n \pmod{\ell^r}.$$

Proof of Theorem 2 (cont.)

Induction

$$\sum_{n=0}^{\infty} p_\alpha(\ell^i n - k) q^n \equiv (q;q)_\infty^{\ell^{r-i} u} \sum_{n=0}^{\infty} \tau_k(\ell^i n) q^n \pmod{\ell^r}.$$

Proof of Theorem 2 (cont.)

Induction

$$\sum_{n=0}^{\infty} p_\alpha(\ell^i n - k) q^n \equiv (q;q)_\infty^{\ell^{r-i}u} \sum_{n=0}^{\infty} \tau_k(\ell^i n) q^n \pmod{\ell^r}.$$

ℓ-non-ordinarity extends

Normalized eigenforms in S_{12k} are ℓ-non-ordinary, hence Δ^k as well

$$\implies \tau_k(\ell^v n) \equiv 0 \pmod{\ell^v}$$

$$\implies \sum_{n=0}^{\infty} p_\alpha(\ell^v n - k) q^n \equiv 0 \pmod{\ell^v}.$$

Example of Theorem 2

Congruences for powers of primes

- $\ell = 17$ is good for $\alpha = 57/61$ with parameter $k = 3$ because

$$17 \mid (24 \cdot 3 - 57/61),$$
$$16 \mid (12 \cdot 3 - 4),$$
$$17 \nmid N_{36}(\mathcal{D}_{36}).$$

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- Check $\operatorname{ord}_{17}(24 \cdot 3 - \frac{57}{61}) = 2$, so our theorem gives that for all n,

$$p_{\frac{57}{61}}(17n - 3) \equiv 0 \pmod{17},$$
$$p_{\frac{57}{61}}(17^2 n - 3) \equiv 0 \pmod{17^2},$$

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- Check $\mathrm{ord}_{17}(24 \cdot 3 - \frac{57}{61}) = 2$, so our theorem gives that for all n,

$$p_{\frac{57}{61}}(17n - 3) \equiv 0 \pmod{17},$$
$$p_{\frac{57}{61}}(17^2 n - 3) \equiv 0 \pmod{17^2},$$
$$p_{\frac{57}{61}}(17^3 n - 3) \not\equiv 0 \pmod{17^3}.$$

Ramanujan's Θ-operator

Definition

We collect all modular forms modulo ℓ of weight k into the space

$$M_{k,\ell} := \{f \pmod{\ell} : f \in M_k \cap \mathbb{Z}[[q]]\}.$$

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Example (Repeated applications of the Θ-operator)

Let $f = \sum_n a_n q^n \in \mathbb{Z}[[q]]$. By Fermat's Little Theorem, we have

$$\Theta^\ell(f) = \sum_n n^\ell a_n q^n \equiv \sum_n n a_n q^n = \Theta(f) \pmod{\ell}.$$

Serre filtration

Definition

For $f \in M_k \cap \mathbb{Z}[[q]]$, define the *filtration of* f *modulo* ℓ by

$$\omega_\ell(f) := \inf\{k \in \mathbb{Z} : f \pmod{\ell} \in M_{k,\ell}\}.$$

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Example (Filtration of Eisenstein series)

The normalized Eisenstein series of weight $\ell - 1$ has Fourier expansion

$$E_{\ell-1}(z) = 1 - \frac{2(\ell-1)}{B_{\ell-1}} \sum_{n=1}^{\infty} \sigma_{\ell-2}(n) q^n \equiv 1 \pmod{\ell}$$

by the Von Staudt-Clausen theorem on divisibility of Bernoulli numbers. Therefore, $\omega_\ell(E_{\ell-1}) = 0$.

Filtration and the Θ-operator

Filtration Lemma

If $\ell \geq 5$ and $f \in M_k \cap \mathbb{Z}[[q]]$, then $\Theta(f) \pmod{\ell}$ is the reduction of a modular form modulo ℓ. Moreover,

$$\omega_\ell(\Theta f) = \omega_\ell(f) + (\ell + 1) - s(\ell - 1)$$

for some integer $s \geq 0$, with equality if and only if $\ell \nmid \omega_\ell(f)$.

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Example

Let $\ell = 5$ and repeatedly apply the Θ-operator to the Delta-function.

Form	Δ	$\Theta(\Delta)$	$\Theta^2(\Delta)$	$\Theta^3(\Delta)$	$\Theta^4(\Delta)$	$\Theta^5(\Delta)$
ω_ℓ	12	18	24	30	12	18

Which arithmetic progressions have congruences?

Theorem 1 (BCC)

Let $\alpha = a/b$. Let $\ell \geq 5$ not divide b such that $\alpha \not\equiv 1, 3 \pmod{\ell}$. If

$$p_\alpha(\ell n + c) \equiv 0 \pmod{\ell}$$

for all n, then $24c \equiv -\alpha \pmod{\ell}$.

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"Key Ingredient" (Kiming-Olsson, 1992)

Let $\ell \geq 5$ and let $k \geq 1$ such that $24k \not\equiv 1, 3 \pmod{\ell}$. If

$$\Theta^{\ell-1}(q^{-s}\Delta^k) \equiv q^{-s}\Delta^k \pmod{\ell}$$

for some integer s, then $s \equiv 0 \pmod{\ell}$.

Proof of Theorem 1

Rewrite in terms of Ramanujan Δ-function.

Write $\alpha = 24k + \ell u$ for some $k \geq 1$ and $u \in \mathbb{Z}_{(\ell)}$. Then

$$\sum_{n=0}^{\infty} p_\alpha(n) q^{n+k} = q^k (q;q)_\infty^{24k+\ell u} \equiv \Delta^k (q^\ell; q^\ell)_\infty^u \pmod{\ell}.$$

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Introduce Θ-operator.

Write $\Delta^k =: \sum_{n=0}^{\infty} \tau_k(n) q^n$ and extract terms of the form $q^{\ell n + c + k}$:

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$$\text{“key ingredient”} \implies 0 \equiv c + k \equiv \tfrac{1}{24}(24c + \alpha) \pmod{\ell}.$$

Which primes ℓ give a congruence?

Theorem 3 (BCC)

Let α be an even integer < 0 or an odd integer > 3. If

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for all n, then $\ell \leq |\alpha| + 4$. In particular, p_α admits finitely many ℓ-balanced congruences.

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"Preparation"

If $\ell \geq 5$ and δ_ℓ is a positive integer, then for any $m \geq 0$ we have

$$\omega_\ell(\Theta^m \Delta^{\delta_\ell}) \geq \omega_\ell(\Delta^{\delta_\ell}) = 12\delta_\ell.$$

Proof of Theorem 3

Rewrite in terms of Θ-operator.

Suppose for contradiction that for some $\ell > |\alpha| + 4$, we have

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By Fermat's little theorem, we conclude that

$$\Theta^{\ell-1}(\Delta^{\delta_\ell}) \equiv \Delta^{\delta_\ell} \pmod{\ell}.$$

Proof of Theorem 3 (cont.)

Study the sequence of filtrations $\omega_\ell(\Theta^i(\Delta^{\delta_\ell}))$

If $0 \leq c < \ell$ satisfies $c \equiv -12\delta_\ell \pmod{\ell}$, then

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Because α is an even integer < 0 or an odd integer > 3, we know that

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Therefore $\omega_\ell(\Theta^{c+1}(\Delta^{\delta_\ell})) < \omega_\ell(\Delta^{\delta_\ell})$, contradicting the "preparation".

Extension of Theorem 3 to rational α?

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Theorem 4 (BCC)

Suppose α is not an even integer ≥ 0. If p_α admits an ℓ-balanced congruence for $\ell \geq |a| + 5b$, then

$$\Psi_{2b}\left(\frac{a}{\ell}\right) \geq b.$$

Illustration of Theorem 4

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Question

Given ℓ, are there restrictions that govern ℓ-balanced congruences?

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Theorem 2 (BCC)

If ℓ is good for α with parameter k and $v \leq \mathrm{ord}_\ell(24k - \alpha)$ is a positive integer, then for all n, we have

$$p_\alpha(\ell^v n - k) \equiv 0 \pmod{\ell^v}.$$

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Remark

Half of primes cannot be the modulus of a balanced congruence for p_α.