All-Pay Auctions with Different Forfeit Functions

Ben Kang

Thomas Jefferson High School for Science and Technology

Mentor: James Unwin, University of Illinois

MIT PRIMES

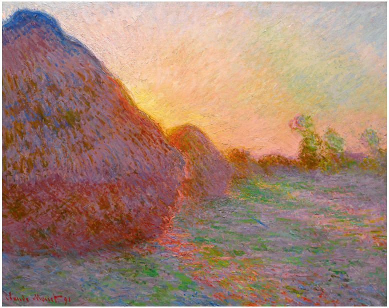
May 19th, 2019

Types of Auctions

- First-price auction
- Second-price auction
- English auction
- Dutch auction
- All-pay auction

Importance

- Commonly used to sell items
- Wars and animal conflicts
- Rent-seeking activities
- Competition with sunk investments
- Mineral rights



Modeling a General Auction

- $X = (X_1, X_2, \ldots, X_n)$ denotes information known to each bidder
- $S = (S_1, S_2, \ldots, S_m)$ denotes additional information known by all

Modeling a General Auction

- $X = (X_1, X_2, \ldots, X_n)$ denotes information known to each bidder
- $S = (S_1, S_2, \ldots, S_m)$ denotes additional information known by all
- $f(s, x)$ is the joint probability distribution of variables satisfying the affiliation inequality

Modeling a General Auction

- $X = (X_1, X_2, \ldots, X_n)$ denotes information known to each bidder
- $S = (S_1, S_2, \ldots, S_m)$ denotes additional information known by all
- $f(s, x)$ is the joint probability distribution of variables satisfying the affiliation inequality
- $Y_1 = \max\{X_j\}_{j \neq 1}$

Modeling a General Auction

- $X = (X_1, X_2, \ldots, X_n)$ denotes information known to each bidder
- $S = (S_1, S_2, \ldots, S_m)$ denotes additional information known by all
- $f(s, x)$ is the joint probability distribution of variables satisfying the affiliation inequality
- $Y_1 = \max\{X_j\}_{j \neq 1}$
- $f_{Y_1}(\cdot|x)$ is the conditional density of Y_1 if $X_1 = x$
- $F_{Y_1}(\cdot|x)$ is the corresponding cumulative distribution

Modeling a General Auction

- $X = (X_1, X_2, \ldots, X_n)$ denotes information known to each bidder
- $S = (S_1, S_2, \ldots, S_m)$ denotes additional information known by all
- $f(s, x)$ is the joint probability distribution of variables satisfying the affiliation inequality
- $Y_1 = \max\{X_j\}_{j \neq 1}$
- $f_{Y_1}(\cdot|x)$ is the conditional density of Y_1 if $X_1 = x$
- $F_{Y_1}(\cdot|x)$ is the corresponding cumulative distribution
- $v(x, y)$ is the expected value of the object to the bidder if $X_1 = x$ and $Y_1 = y$

Payoff Function

$$W_i = \begin{cases} V_i - b_i & b_i > \max_{j \neq i} b_j \\ -b_i & b_i < \max_{j \neq i} b_j \\ \frac{V_i}{\#\{k : b_k = b_i\}} - b_i & b_i = \max_{j \neq i} b_j \end{cases}$$

- V_i denotes the value of the object to the bidder
- b_i denotes the value of the bid

Expected Payoff

$$\Pi(b, x) = \int_{-\infty}^{\alpha^{-1}(b)} v(x, y) f_{Y_1}(y|x) dy - b$$

All-Pay Auctions

- Maximize payoff
- Bidder 1 also follows bidding strategy

Bidding Condition

$$v(x, \alpha^{-1}(b)) f_{Y_1}(\alpha^{-1}(b)|x) \frac{1}{\alpha'(\alpha^{-1}(b))} - 1 = 0$$

Bidding Strategy

$$\alpha(x) = \int_{-\infty}^{x} v(t, t) f_{Y_1}(t|t) dt$$

First-Price and All-Pay Auctions

All-Pay Bidding Equilibrium

$\alpha(x) = \int_{-\infty}^{x} v(t,t) f_{Y_1}(t|t) dt$

First-Price Bidding Equilibrium

$\alpha(x) = \int_{-\infty}^{x} v(s,s) \frac{f_{Y_1}(s|s)}{F_{Y_1}(s|s)} \exp\left(-\int_{s}^{x} \frac{f_{Y_1}(t|t)}{F_{Y_1}(t|t)} dt\right) ds$

Theorem (Krishna and Morgan)

All-pay auctions earn at least as much revenue as first-price auctions.

Constant Entrance Fee

Fee not Returned to Winner

$$W_i = \begin{cases} V_i - b_i - c & b_i > \max_{j \neq i} b_j \\ -b_i - c & b_i < \max_{j \neq i} b_j \\ \frac{V_i}{\#\{k : b_k = b_i\}} - b_i - c & b_i = \max_{j \neq i} b_j \end{cases}$$

Bidding Condition

$$v(x, \alpha^{-1}(b)) f_{Y_1}(\alpha^{-1}(b)|x) \frac{1}{\alpha'(\alpha^{-1}(b))} - 1 = 0$$

- Does not affect bidding strategy

Constant Entrance Fee

Fee Returned to Winner

$$W_i = \begin{cases} V_i - b_i & b_i > \max_{j \neq i} b_j \\ -b_i - c & b_i < \max_{j \neq i} b_j \\ \frac{V_i}{\#\{k : b_k = b_i\}} - b_i - c & b_i = \max_{j \neq i} b_j \end{cases}$$

Bidding Strategy

$$\alpha(x) = \int_{-\infty}^{x} (v(t,t) + c) f_{Y_1}(t|t) dt$$

- Asymmetric entrance fee increases bidding strategy

Fractional Forfeit

Payoff Function

$$W_i = \begin{cases} V_i - b_i & b_i > \max_{j \neq i} b_j \\ -\beta b_i & b_i < \max_{j \neq i} b_j \\ \frac{V_i}{\#\{k : b_k = b_i\}} - b_i & b_i = \max_{j \neq i} b_j \end{cases}$$

- $\beta \in [0, 1]$ is any real number

Fractional Forfeit

Bidding Strategy

$$\alpha(x) = \int_{-\infty}^{x} v(s,s) \frac{f_{Y_1}(s|s)}{\beta+(1-\beta)F_{Y_1}(s|s)} \exp\left(-(1-\beta)\int_{s}^{x} \frac{f_{Y_1}(t|t)}{\beta+(1-\beta)F_{Y_1}(t|t)} dt\right) ds$$

Theorem (Revenue Comparison)

The revenue generated for any auction where $\beta \in [0,1]$ is at most the revenue when $\beta = 1$

Exponential Forfeit

Payoff Function

$$W_i = \begin{cases} V_i - b_i & b_i > \max_{j \neq i} b_j \\ -e^{b_i} & b_i < \max_{j \neq i} b_j \\ \frac{V_i}{\#\{k : b_k = b_i\}} - b_i & b_i = \max_{j \neq i} b_j \end{cases}$$

Bidding Strategy

$\alpha(x) \approx \int_{-\infty}^{x} \mathrm{d}t \left(\frac{f_{Y_1}(t|t)}{1 - F_{Y_1}(t|t)} \right)$ when the bid is large.

- Does not depend on the value of the object to the bidder

Summary

We examined results of all-pay auctions with the following different forfeit functions:

- Constant Entrance Fee
- Fractional Forfeit
- Exponential Forfeit

Future Work

In the future, we will analyze more types of all-pay auctions that have:

- More forfeit functions
- Risk-averse bidders
- Multiple prizes

Key References

V. Krishna and J. Morgan, *An analysis of the war of attrition and the all-pay auction.* Journal of Economic Theory 72.2 (1997): 343-362.

P. Milgrom and R. Weber, *A theory of auctions and competitive bidding.* Econometrica: Journal of the Econometric Society (1982): 1089-1122.

Aknowledgements

I am grateful to the **MIT PRIMES-USA** faculty for the opportunity to present at this meeting, my mentor, **Prof. James Unwin**, for facilitating my research, and the head mentor, **Dr. Tanya Khovanova**, for providing feedback.

Thank You!