Minimal Percolating Sets with Time-Dependent Bootstrap Percolation

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Introduction

Idea

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  - Orientational ordering processes of magnetic alloys
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  - Modeling infectious diseases

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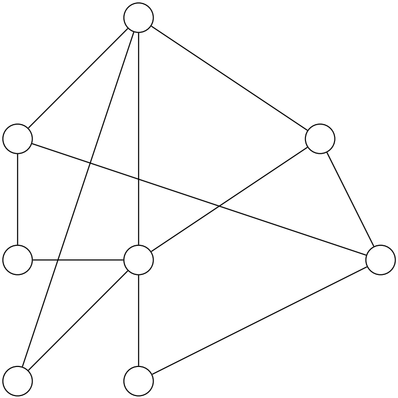
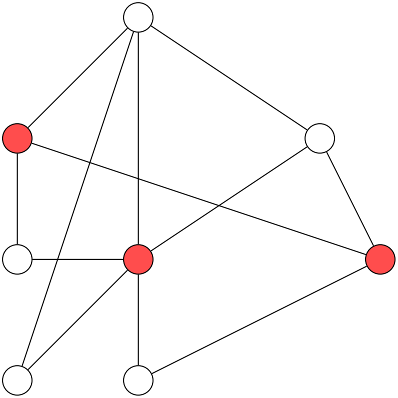
  - Failure of units in structured collection of computer memory
  - Modeling infectious diseases
- Advantages when used to model infectious diseases
  - Individual behavior
  - Spatial aspects
  - Mixing of individuals

Bootstrap percolation: an example



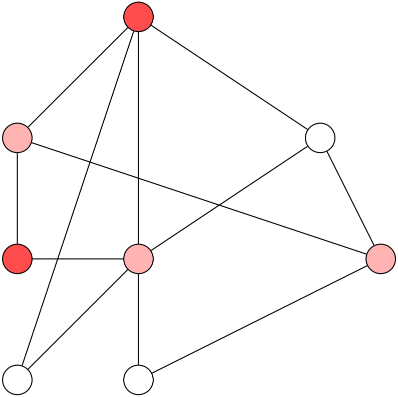
Bootstrap percolation: an example

$t = 0$
$r = 2$



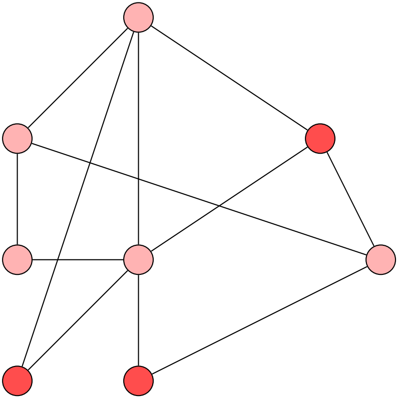
Bootstrap percolation: an example

$t = 1$
$r = 2$



Bootstrap percolation: an example

$t = 2$
$r = 2$



r-Bootstrap percolation: definition

Definition

Suppose we have a graph $G(V, E)$, some positive integer r, and a set $A_0 \subset V$ of initially infected nodes. Then in the $r-$bootstrap percolation process, the set of infected vertices at time $t+1$ is given by

$$A_{t+1} = A_t \cup \{v \in V(G) : |N(v) \cap A_t| \geq r\}$$

where $N(v)$ is the set of adjacent vertices to v.

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Notation

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A percolating set A is **minimal** if for any $v \in A$,

$$P(A \setminus \{v\}) \neq V.$$

Example: Minimal Percolating Set

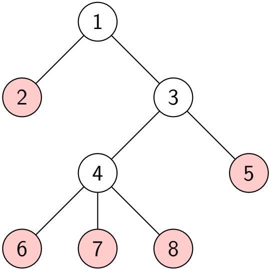

Figure: In this tree, having nodes $2, 5, 6, 7, 8$ initially infected is sufficient to ensure that the whole tree is infected at some finite time for $r = 2$. Further, since each of them has only 1 neighbor, removing any will cause the remaining set to not percolate.

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Motivation

Rate of disease spread is not constant in nature.

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Definition

Consider a graph $G(V, E)$, a function $F(t)$, and a set $A_0 \subset V$ of initially infected nodes. Then in the $F(t)-$bootstrap percolation process the set of infected vertices at time $t+1$ is

$$A_{t+1} = A_t \cup \{v \in V(G) : |N(v) \cap A_t| \geq \lceil F(t) \rceil\}.$$

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Let v be a node that is not isolated. We denote by $t_p(v)$ the largest time t at which $F(t)$ is not larger than the number of infected neighbors of v at t.

Algorithm for $F(t)$-BP on $T(V, E)$

- **Step 1. Initialize tree:** Let $A_0 = \varnothing$. For $v \in V$ set $t_f(v)$ arbitrarily large, and set it to true for needing to be considered.
- **Step 2. Percolate using current A_0.** Stop the algorithm if $P(A_0) = V$: the resulting A_0 is a smallest minimal percolating set.
- **Step 3. Infection.** For an unconsidered v furthest away from the root (isolated first).
  - if v is isolated, add v to A_0.
  - otherwise, set t_f of the parent of v to $t_p(v) - 1$ if it is smaller than the current t_f of the parent.
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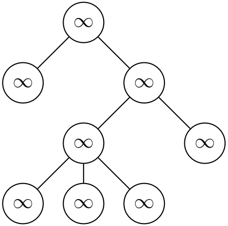

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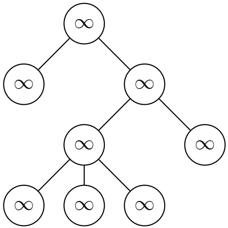

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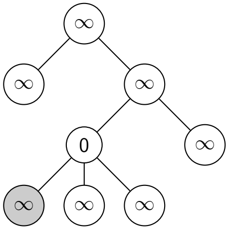

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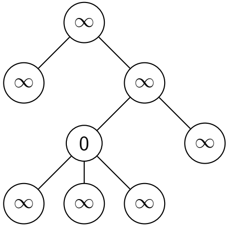

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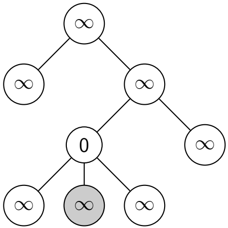

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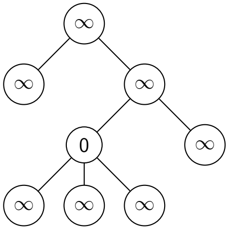

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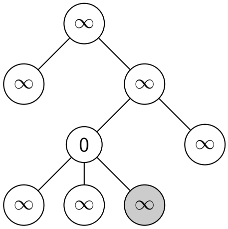

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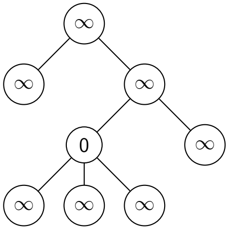

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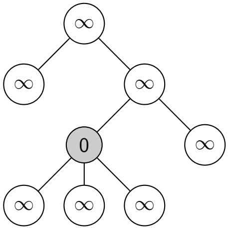

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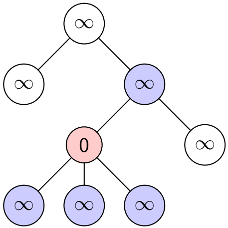

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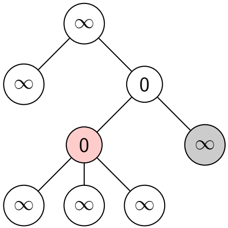

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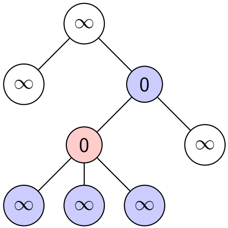

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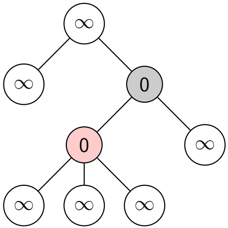

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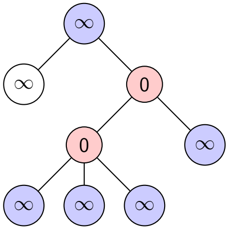

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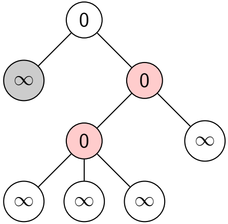

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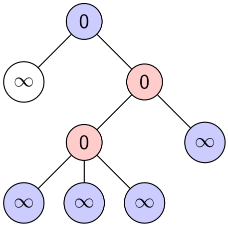

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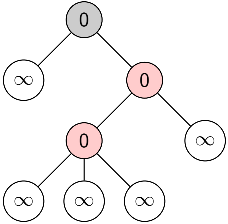

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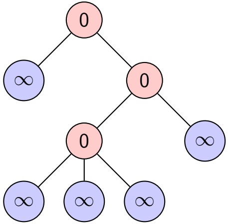

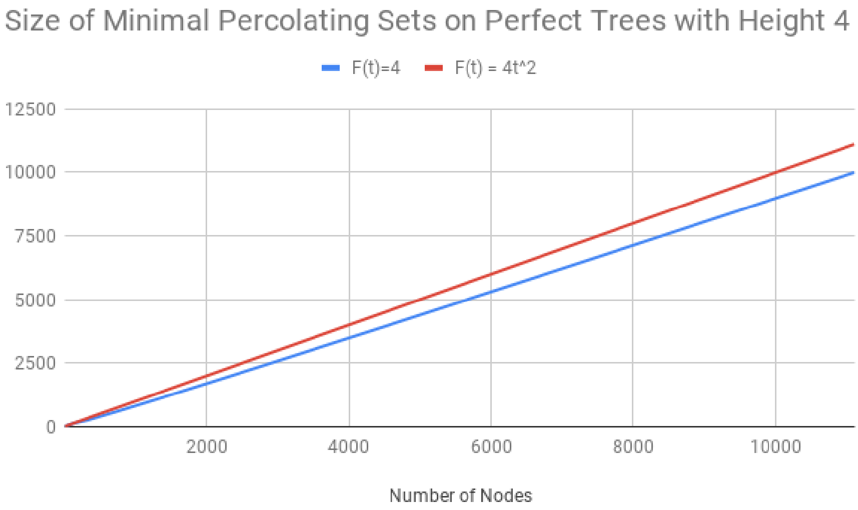

Figure: The size of smallest minimal percolating sets on perfect trees with height 4, with a constant and a non-constant percolation function

Future Directions

- Devise a similar algorithm for largest minimal percolating set.
- Study the size of largest and smallest minimal percolating sets in specific trees and lattices, with specific types of percolation functions.

Acknowledgements

- Dr. Laura Schaposnik
- Dr. Tanya Khovanova
- My parents
- MIT PRIMES program