On the degenerate Turán problem and its variants

Sunay Joshi

Mentor: Dr. Zilin Jiang

Bergen County Academies

May 18-19, 2019

MIT PRIMES Conference

Graphs

- A *graph* $G = (V, E)$ consists of vertices V and edges E.
- A *subgraph* $H = (V', E')$ of G satisfies $V' \subseteq V$ and $E' \subseteq E$.
- A *bipartite graph* G consists of two independent sets of vertices.

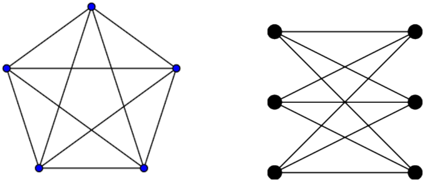

Figure: K_5 and $K_{3,3}$ [1, 2]

The extremal number

- Given a family $\mathcal{F}$ of graphs, $\mathrm{ex}(n, \mathcal{F})$ is max number of edges in n-vertex graph that does not contain any element of $\mathcal{F}$.
- We write $\mathrm{ex}(n, F)$ if $\mathcal{F} = \{F\}$.

Example: $\mathrm{ex}(6, K_3) = 9$. (Try adding another edge.)

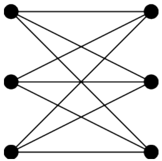

Figure: $K_{3,3}$, again [2]

History of the Turán problem

- Mantel, 1907: $\mathrm{ex}(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$.
- Turán, 1941: $\mathrm{ex}(n, K_{r+1}) = \left\lfloor \frac{r-1}{r} \cdot \frac{n^2}{2} \right\rfloor$.
- Erdős–Stone–Simonovits, 1966:

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right) \binom{n}{2} + o(n^2),$$

  where $\chi(F)$ is the *chromatic number* of F.
- If $\chi(F) = 2$, $\mathrm{ex}(n, F) = o(n^2)$... unsatisfactory!

The degenerate Turán problem

To determine the asymptotic behavior of $\mathrm{ex}(n, \mathcal{F})$ for $\mathcal{F}$ consisting of bipartite graphs is the *degenerate Turán problem.*

Conjecture (Rational Turán exponents)

For every rational $r \in [1, 2]$, there exists a graph F with $\mathrm{ex}(n, F) = \Theta(n^r)$.

Bukh and Conlon's breakthrough

Theorem (Bukh–Conlon, 2017)

For every rational $r \in [1, 2]$, there exists a finite family $\mathcal{F}$ of graphs with $\mathrm{ex}(n, \mathcal{F}) = \Theta(n^r)$.

- The lower bound for Conjecture 1 follows as a corollary.

Question: What was Bukh and Conlon's construction for $\mathcal{F}$?

Blowups of rooted graphs

- A *rooted graph* F consists of a nonempty set of roots R and is denoted (F, R).
- The *blowup* $\mathcal{F}^p$ of (F, R) consists of all possible unions of p copies of F sharing roots.

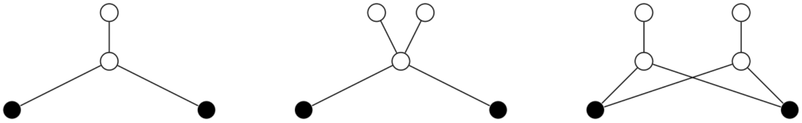

Figure: The 3-star F and two elements of $\mathcal{F}^2$

Bukh and Conlon's construction

Theorem (Bukh–Conlon, 2017)

Let (F, R) be a *balanced* rooted graph with a unrooted vertices and b edges in F. Then for sufficiently large p, $\mathrm{ex}(n, \mathcal{F}^p) = \Theta(n^{2-a/b})$.

- Proof idea: combinatorics + Lang-Weil bound yield upper bound on expected number of copies of F with fixed roots.

Main Results

A *closed rooted graph* has all of its leaves as roots.

Proposition (Classifying balanced rooted graphs)

Let (G, R) be a closed rooted graph. Then (G, R) is balanced if and only if every closed rooted subgraph of G has density at most that of G.

Main Results

Let $H_{s,t}$ be the graph formed by matching corresponding vertices in two vertex-disjoint copies of $K_{s,t}$.

Theorem (Jiang–Ma–Yepremyan, 2018)

For all $t \geq s \geq 2$, $\text{ex}(n, H_{s,t}) = O(n^{2-2/(2s+1)})$.

We present a new proof using the Erdős–Simonovits Reduction Theorem.

Main Results

We can define the *asymmetric Turán number* $\mathrm{ex}(m, n, \mathcal{F})$ similarly.

Proposition (Lower bound for F^p)

Let (F, R) be a balanced rooted bipartite graph with a unrooted vertices and b edges, and let $m \leq n$. Then for sufficiently large p,

$$\mathrm{ex}(m, n, F^p) = \Omega(mn^{1-a/b}).$$

Proposition (Lower bound for theta graphs)

Let $k \geq 1$, and let $q > \binom{6k^2}{2}$ be a prime power. Let $m = q^t$ and $n = q^{2k-2-\frac{k-2}{k}t}$, where $t \leq k$. Then for sufficiently large p,

$$\mathrm{ex}(m, n, \theta_{k,p}) = \Omega(m^{\frac{k+2}{2k}} n^{\frac{1}{2}}).$$

Future Work

- Find more applications of the Erdős–Simonovits Reduction Theorem.
- Find stronger lower bounds on $\mathrm{ex}(m, n, F^p)$ for $m \ll n$.
- Prove lower bound on $\mathrm{ex}(m, n, \theta_{k,p})$ for all m, n.

Acknowledgements

I would like to thank:

- My mentor, Dr. Zilin Jiang
- Dr. Tanya Khovanova
- MIT PRIMES-USA Program
- My family

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