Analysis of the One Line Factoring Algorithm on Large Semiprimes

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Introduction

What is a factoring algorithm?

Find a divisor of N.

The Naïve Algorithm – Trial Division

Factoring N

Divide by every prime in $[1 \ldots \sqrt{N}]$

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Example: $N = 119$

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$119/2$ not an integer.

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$119/5$ not an integer.

$119/7 = 17$ is an integer!

Simple Factoring Algorithm: Fermat

Factoring N

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Factoring N

- Let $a := \lceil \sqrt{N} \rceil$
- Let $b := a^2 - N$
- Repeat until b is a square:
  Increase a by 1 $(a := a + 1)$
  $b := a^2 - N$
- When b is a square, then
  $(a - \sqrt{b})$ is a factor.

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- $a := \lceil \sqrt{119} \rceil = 11$

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- $a := \lceil \sqrt{119} \rceil = 11$
- $b := 11^2 - 119 = 2$

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- $a := \lceil \sqrt{119} \rceil = 11$
- $b := 11^2 - 119 = 2$
- b (2) is not a square:
  $a := a + 1 = 12$
  $b := 12^2 - 119 = 25$

Simple Factoring Algorithm: Fermat

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- $b = 25$ is a square, so
  $12 - \sqrt{25} = 7$ is a factor.

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Works because of square difference $x^2 - y^2 = (x + y)(x - y)$

One Line Factoring Algorithm?

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Slower than the leading algorithms

One Line Factoring Algorithm?

Slower than the leading algorithms

Much less space required

“One Line” O.o

One line of PARI/GP…

```
OLF(x)=;i=1;while(i<x,if(issquare(ceil(sqrt(i*x)
)^2%x),return(gcd(x,floor(ceil(sqrt(i*x))-sqrt((
ceil(i*x))^2)%))))));i++)
```

The One Line Factoring Algorithm

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Repeat for $k = 1$ to $k = N$:

- Let $m := \left\lceil \sqrt{N \cdot k} \right\rceil^2 \% \ N$
- If m is a square:
  Factor is
  $\mathrm{GCD}(N, \left\lceil \sqrt{N \cdot k} \right\rceil - \sqrt{m})$

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- When $k = 1$, $m = 2$
- When $k = 2$, $m = 18$

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Example: $N = 119$

- When $k = 1$, $m = 2$
- When $k = 2$, $m = 18$
- When $k = 3$, $m = 4$

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- When $k = 3$, $m = 4$
- Factor:
  $\mathrm{GCD}(119, \left\lceil \sqrt{119 \cdot 3} \right\rceil - \sqrt{4})$
  $\mathrm{GCD}(119, 17) = 17$

Semiprimes

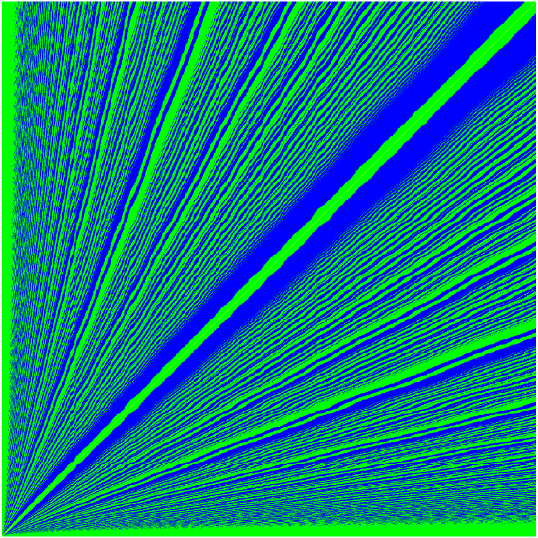
Factor numbers $N = pq$

Semiprimes

Factor numbers $N = pq$

Applications in cryptography (like RSA)

Pretty Picture: Result of factoring pq



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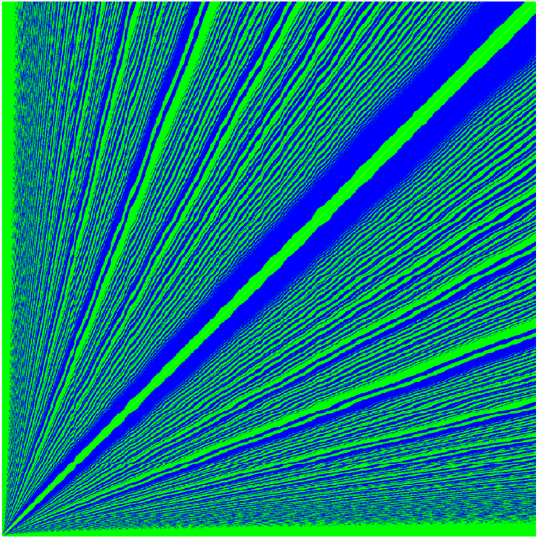
Factoring pq:



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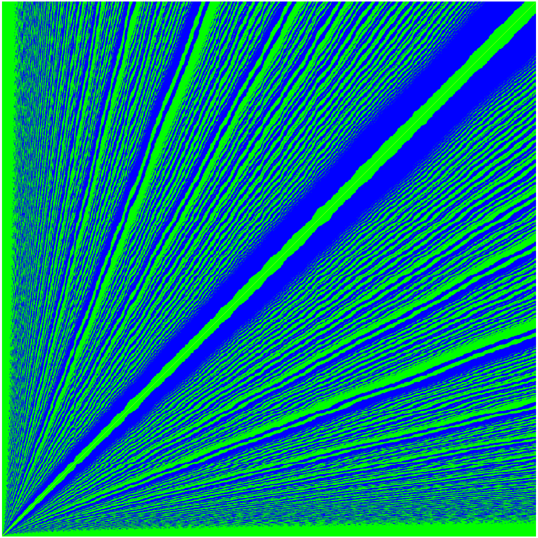
Factoring pq:

- X-coordinate is prime p,

  Y-coordinate is prime q,

  p, q are first 1600 primes



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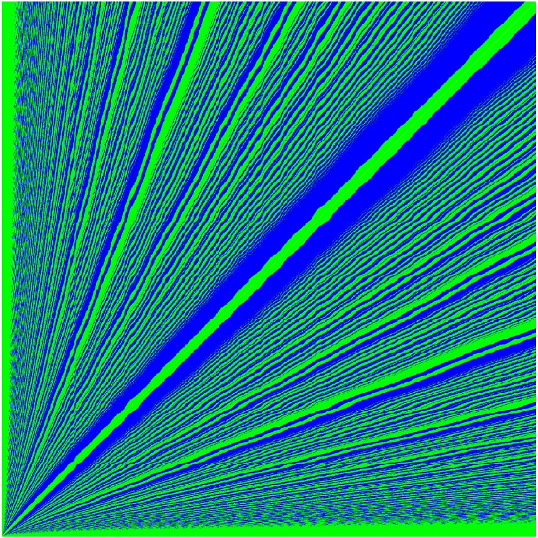

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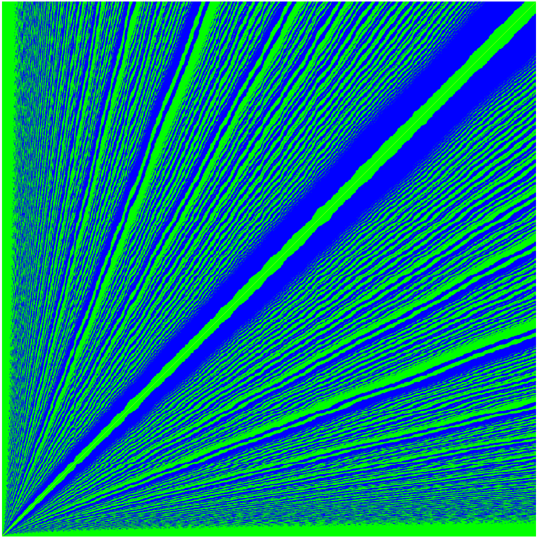

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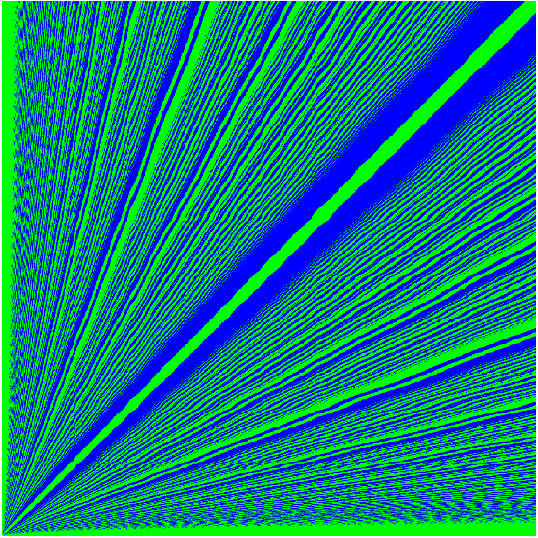
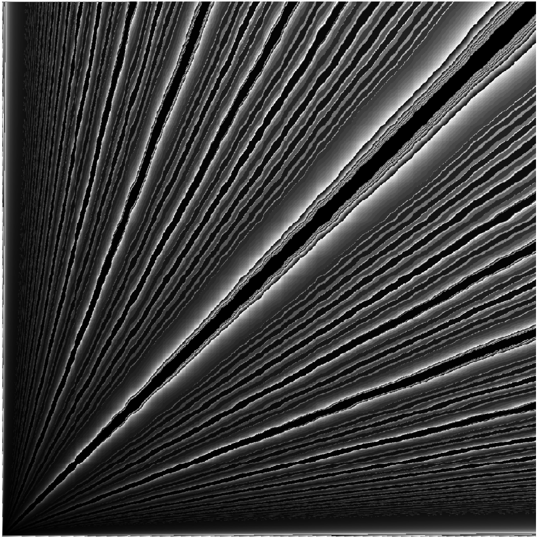
- If p,q are close: Smaller prime returned
- Probability of green is $\sim 50\%$



Performance of OLF on semiprimes



Performance of OLF on semiprimes

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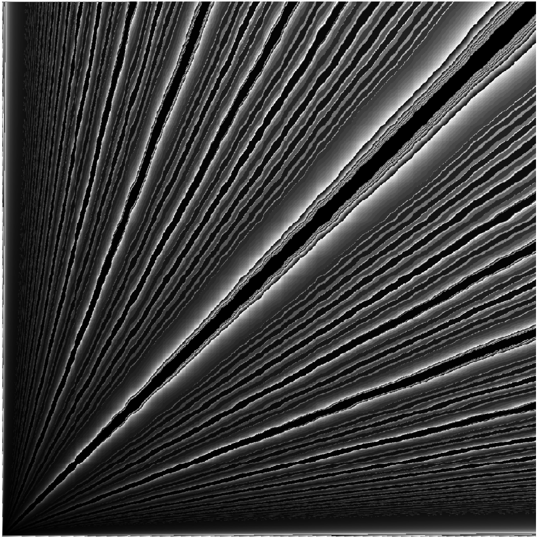

Performance of OLF on semiprimes

Number of iterations to factor pq:

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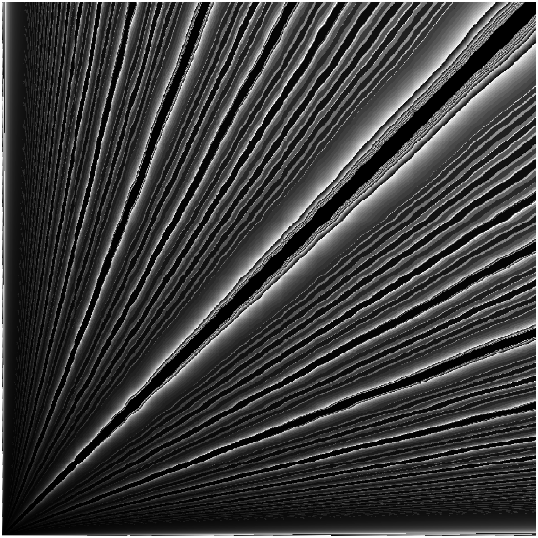
- Points colored from black to white; Whiter means more iterations required



Improving Efficiency: Reduce Iterations?

The algorithm required trying **every number** from $k = 1$ to (at most) $k = N$

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Can we skip some k?

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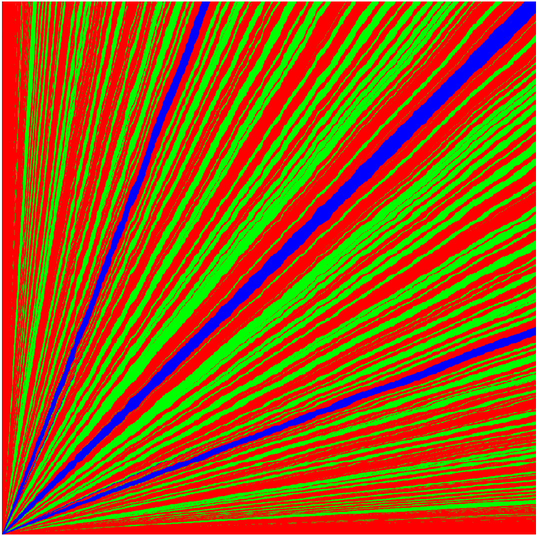
Can we skip some k?

What if we just use squarefree k?

Pretty Picture: Where is squarefree OLF better?



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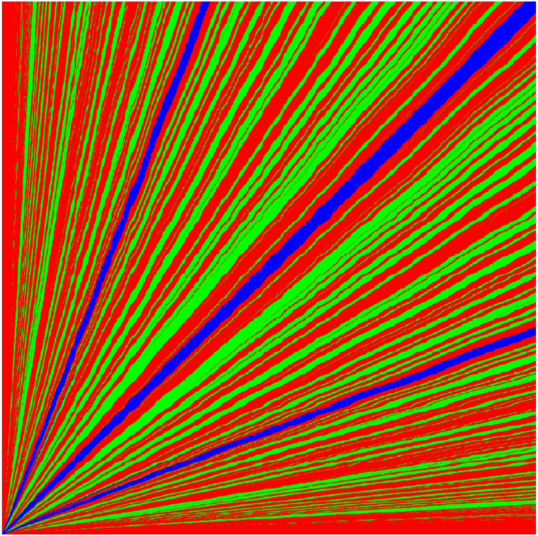
- Green: Squarefree approach faster (fewer iterations)



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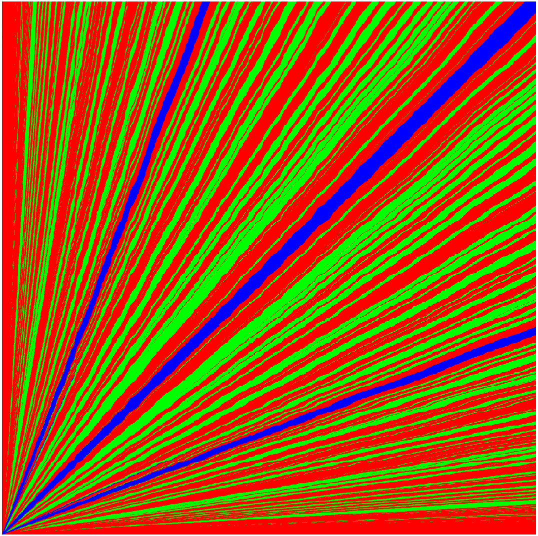
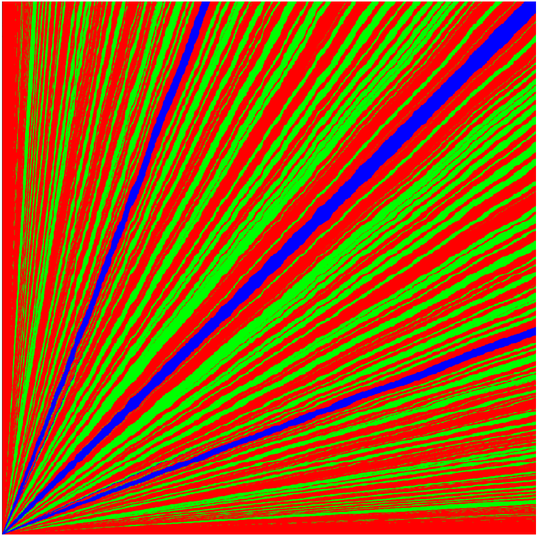
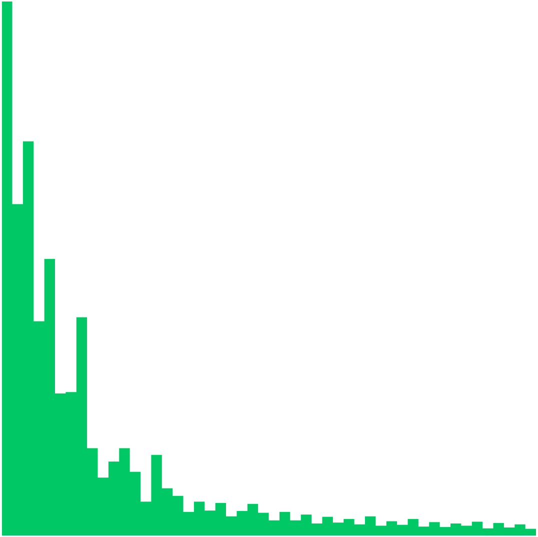


Pretty Picture: Where is squarefree OLF better?

- Green: Squarefree approach faster (fewer iterations)
- Distinct regions where this is more efficient
- Better on roughly $\sim 35.5\%$ of semiprimes



How many iterations to factor a general integer?



- k^{th} bar: Amount of integers that requires k iterations to factor

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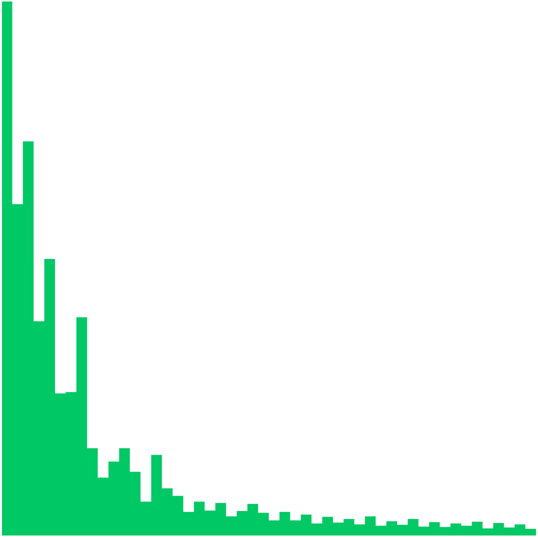

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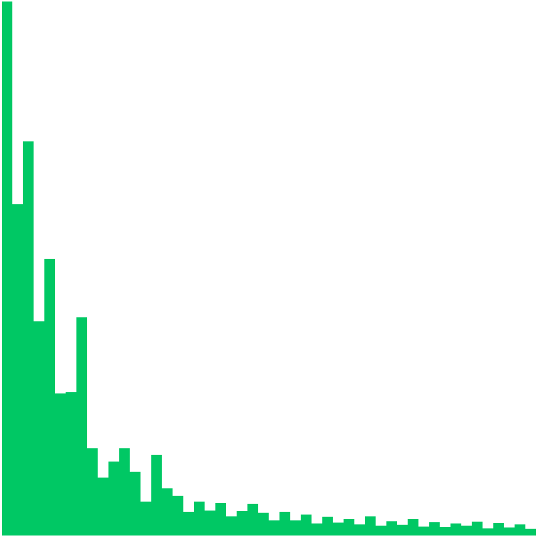
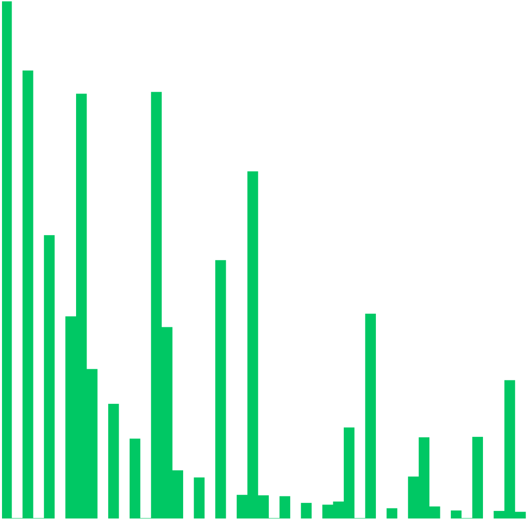


- k^{th} bar: Amount of integers that requires k iterations to factor
- Decreases rapidly
- Therefore, skipping k will not **always** help.

How many iterations to factor semiprimes?



- However, the picture is different if only factoring semiprimes

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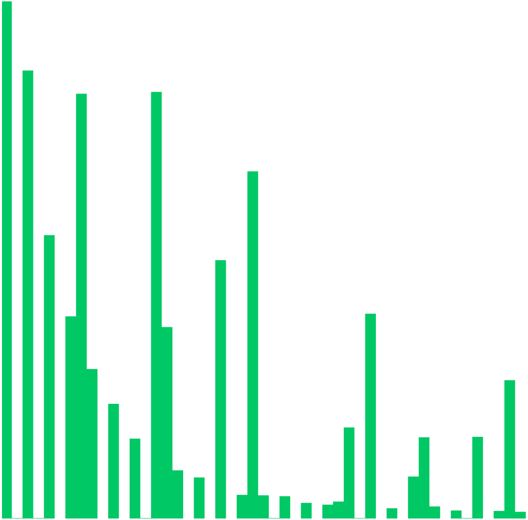

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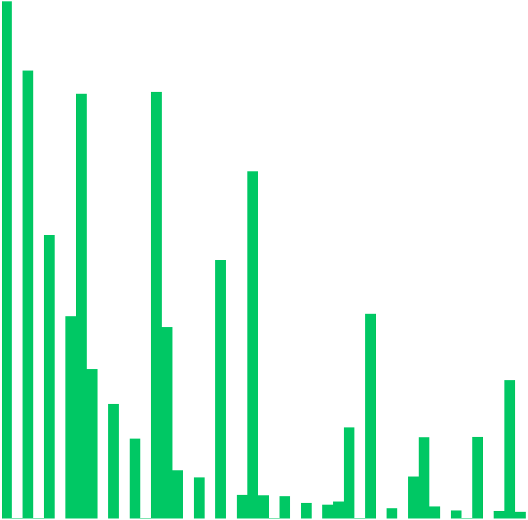

- However, the picture is different if only factoring semiprimes
- Many k not used.
- (Conjecture:) k only has to be $\{0, 1, 3, 5, 7\}$ modulo 8

Further Research

- What causes the strange bands?

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- What causes the strange bands?
- Can we precisely definine when the lower prime is returned?
- Prove the semiprime iterations conjecture.
- When can we skip k in the general algorithm (not just semiprimes)?
- Anything else to make it faster!

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