Critical Lattices of Symmetric Convex Domains

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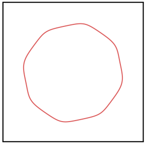
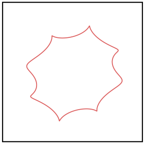
Convex Symmetric Domains in $\mathbb{R}^2$

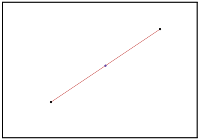
1. Symmetric about the Origin
2. Convex
3. Bounded
4. Nonempty Interior (Positive Area)



Example



Non-example for property 2



Non-example for property 4

Lattices

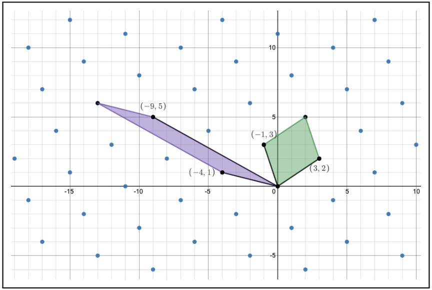
- Lattices in $\mathbb{R}^2$, Λ: The points of Λ with basis $\mathbf{a} = (\alpha_1, \alpha_2)$, $\mathbf{b} = (\beta_1, \beta_2)$ is the set:

$$\{u \cdot \mathbf{a} + v \cdot \mathbf{b} : u, v \in \mathbb{Z}\}.$$

- Covolume, $\mathrm{cov}(\Lambda)$:

$$\mathrm{cov}(\Lambda) = |\det(\mathbf{a}, \mathbf{b})| = |\alpha_1 \cdot \beta_2 - \alpha_2 \cdot \beta_1|.$$

A Lattice with two different bases and their fundamental domains.

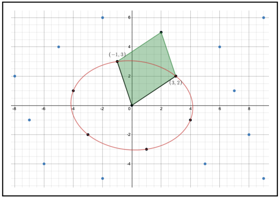


Admissible Lattices

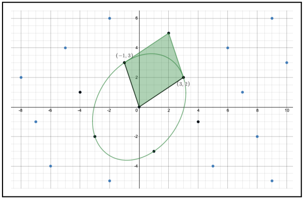
- A lattice Λ is K-admissible if the only point in common with Λ and the interior of K is the origin.
- Λ is critical for K if

$$\mathrm{cov}(\Lambda) = \min_{K\text{-admissible } \Lambda'} \mathrm{cov}(\Lambda').$$

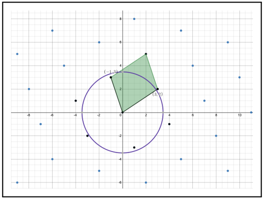
- Critical Value, $\triangle K$: For a K-critical Λ, $\triangle K = \mathrm{cov}(\Lambda)$.



Critical+Admissible



Admissible

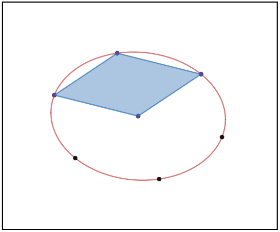


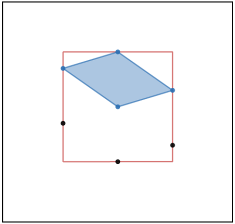
Neither

Parallelograms and Critical Lattices

(Minkowski)

Any critical Λ contains three points P_1, P_2, P_3 on the boundary C of K such that $OP_1P_2P_3$ is a parallelogram.



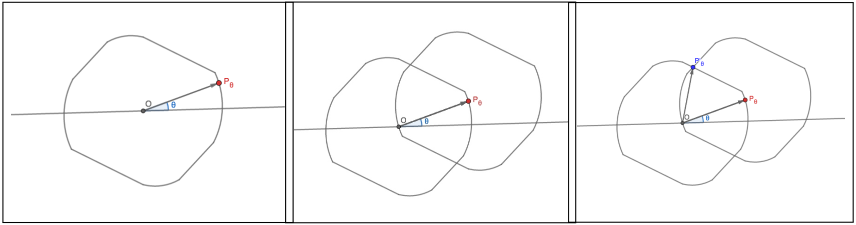


Λ_θ

- Λ_θ : Let P'_θ be any point in $\{C \cap (C + P_\theta)\}$. Then

$$\Lambda_\theta := \langle P_\theta, P'_\theta \rangle.$$

- Λ_θ is always K-admissible.



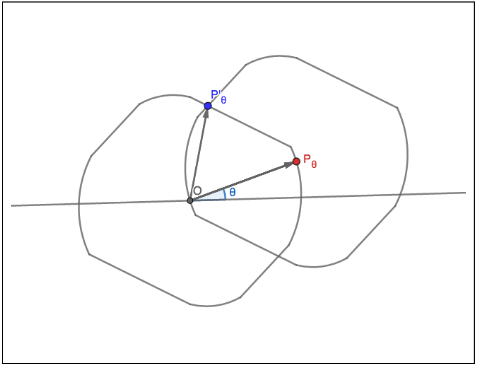
P_θ Translation P'_θ

Λ_θ and Critical Lattices

(Minkowski)

Λ is K-critical if and only if $\Lambda = \Lambda_{\theta_0}$ for some θ_0 and

$$\text{cov}(\Lambda_{\theta_0}) = \min_{0 \le \theta < 2\pi} \text{cov}(\Lambda_\theta).$$

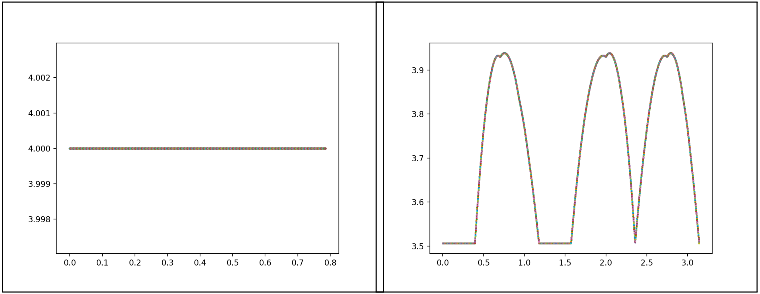


Question #1

Question #1:

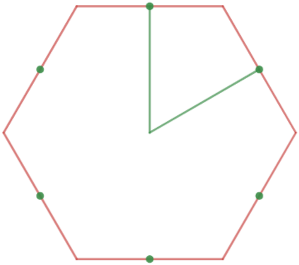
Over all K, what are the possibilities for set $S(K) = \{\theta \in [0, 2\pi] : \text{cov}(\Lambda_\theta) = \triangle K\}$?

- Examples in Literature: Ellipse, Parallelogram, Hexagon
- Using Minkowski's procedure, we calculated $S(K)$ for many other shapes.

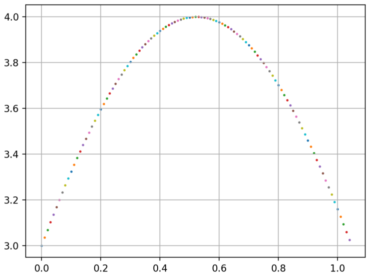


Regular Polygons

When K are regular $6n$-gons, $S(K)$ consists of $6n$ distinct discrete points for n ≥ 1.



Critical lattice for hexagon



$\frac{1}{\text{cov}(\Lambda_\theta)}$ vs θ, $0 \leq \theta \leq \frac{\pi}{3}$ for the hexagon

Result

Theorem (S., R.)

For every closed set $\mathcal{C} \subset [0, \frac{\pi}{3}]$, there exists a K such that

$$S(K) = \bigcup_{i=0}^{6} (\mathcal{C} + \frac{i\pi}{3}).$$

- $S(K)$ is always closed.

Cantor Set

- We considered pathological sets such as the Cantor set.
- Cantor set is the limit of a process that eliminates the middle third in each interval for each iteration.
- We constructed the Cantor Shape in a similar manner

Figure 1: First 7 iterations of Cantor Set

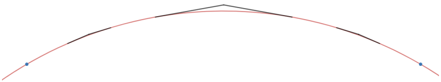
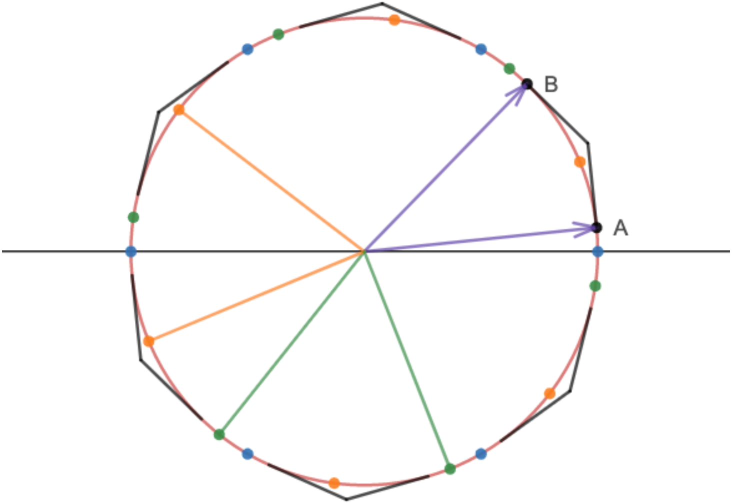


Figure 2: Cantor Shape at the Second Stage

Sketch of Proof: Procedure

- $\mathcal{O} = [0, \frac{\pi}{3}] \setminus \mathcal{C} = \bigcup_{i=0}^{n} (a_i, b_i)$.
- For each interval (a_i, b_i), we create a bump using the tangents at angles a_i and b_i.
- Translate (a_i, b_i) by $\frac{\pi}{3}$ and repeat for each of the other $\frac{1}{6}$-arcs of the shape.

Sketch of Proof: Diagram



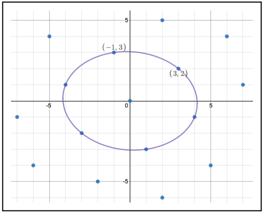
Future Work: Lattice Packings

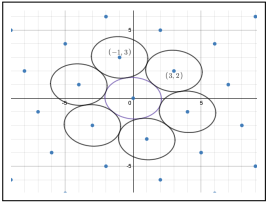
Q(K)

$$Q(K) = \frac{V(K)}{\triangle K}$$

Connection to Packings

Q(K) is called the packing density since it is related to the most economical way to pack the plane with copies of K so that their centers form a lattice.





Future Work: Known Results

Theorem (Minkowski)

$Q(K) \leq 4$

Theorem (Mahler)

$Q(K) \geq \sqrt{12}$

Question #2:

What are all the domains with the worst packing densities?

Acknowledgements

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