Introduction to Representation Theory

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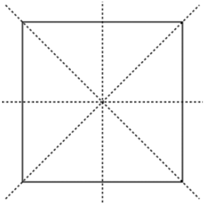
What is a Representation?

Let G be a group. A representation ρ of G with dimension n is a function that assigns every $g \in G$ an $n \times n$ matrix $\rho(g)$, such that

$$\rho(gh) = \rho(g)\rho(h) \text{ (matrix multiplication).}$$

- Equivalently, a representation is a group homomorphism $\rho : G \to GL(n, F)$ (the group of $n \times n$ invertible matrices over F), where F is any field.
- The matrices $\rho(g)$ can be seen as linear transformations acting on an n-dimensional vector space (defined over F).
- In this talk, we will consider finite groups G and take $F = \mathbb{C}$; these representations exhibit nicer properties, as we will see.

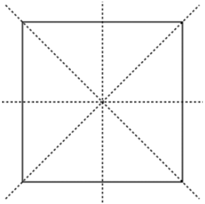
Example: Representation of D_8



Consider the group of symmetries D_8 of the above square, centered at the origin. Because the center is fixed by any symmetry, each element of D_8 corresponds to a 2-dimensional linear transformation, which corresponds to a 2×2 matrix over $\mathbb{R}$ or $\mathbb{C}$.

This leads to a 2-dimensional representation of D_8:

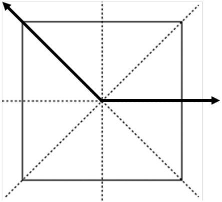
Example: Representation of D_8



Let *id* be the group identity, r correspond to a 90° counterclockwise rotation, and s correspond to a reflection about the x-axis.

$$
\begin{matrix}
id & r & r^2 & r^3 \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\
s & sr & sr^2 & sr^3 \\
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\end{matrix}
$$

Equivalent Representations

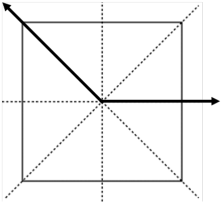


Call two representations ρ_1 and ρ_2 *equivalent* if they have the same dimension and ρ_2 can be obtained from ρ_1 by a change of basis.

- Formally, this means there exists a square matrix T such that for all $g \in G$,
$$\rho_2(g) = T^{-1}\rho_1(g)T.$$

Suppose we change our basis from $\{[1,0],[0,1]\}$ to $\{[1,0][-1,1]\}$, as shown. Then this yields an equivalent representation of D_8:

Equivalent Representations



$$\begin{array}{cccc} id & r & r^2 & r^3 \\ \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} & \begin{bmatrix}1 & -2\\1 & -1\end{bmatrix} & \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix} & \begin{bmatrix}-1 & 2\\-1 & 1\end{bmatrix} \\ s & sr & sr^2 & sr^3 \\ \begin{bmatrix}1 & -2\\0 & -1\end{bmatrix} & \begin{bmatrix}-1 & 0\\-1 & 1\end{bmatrix} & \begin{bmatrix}-1 & 2\\0 & 1\end{bmatrix} & \begin{bmatrix}1 & 0\\1 & -1\end{bmatrix} \end{array}$$

In this case, our matrix $T = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$.

FG-Modules

Let F be a field and G be a group. An FG-module is a vector space V over F with left multiplication by elements of G, such that multiplication by any element g is a linear transformation in V.
There is a correspondence between FG-modules and representations of G:

Theorem

- *For any n-dimensional representation ρ over F of G, F^n is an FG-module if for any $v \in F^n$ and $g \in G$, gv is defined as*

$$gv = \rho(g)v.$$

- *For any n-dimensional FG-module V with basis $\mathscr{B}$, then $\rho : G \to GL(n, F)$ is a representation if for all $g \in G$, we define*

$$\rho(g) = [g]_{\mathscr{B}}.$$

FG-Submodules

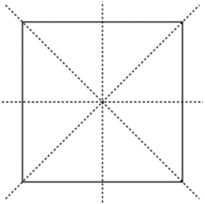
If V is an FG-module, then an FG-submodule of V is any subspace W of V which is also an FG-module.

- Equivalently, W is an FG-submodule of V if for all $g \in G$ and $w \in W$, $gw \in W$ (multiplication in V).

For example, the following representation of D_8 corresponds to a $\mathbb{C}D_8$-module, and has two $\mathbb{C}D_8$-submodules:

$$\begin{matrix} id & r & r^2 & r^3 \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \\ s & sr & sr^2 & sr^3 \\ \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{matrix}$$

FG-Submodules



The two submodules are the subspaces generated by the vectors $[2, -1]$ and $[0, 1]$, respectively. Using the unique basis for each subspace, the corresponding representations of D_8 are:

id	r	r^2	r^3
$[1]$	$[-1]$	$[1]$	$[-1]$
s	sr	sr^2	sr^3
$[1]$	$[-1]$	$[1]$	$[-1]$

id	r	r^2	r^3
$[1]$	$[1]$	$[1]$	$[1]$
s	sr	sr^2	sr^3
$[-1]$	$[-1]$	$[-1]$	$[-1]$

Reducibility of FG-Modules

An nonzero FG-module V is called *reducible* if it has an FG-submodule not equal to $\{0\}$ or V. Otherwise, it is *irreducible*.

- If V is a n-dimensional reducible FG-module with a k-dimensional submodule W, then there exists a basis $\mathscr{B}$ of V such that for all $g \in G$,

$$[g]_{\mathscr{B}} = \left[\begin{array}{c|c} X_g & 0 \\ \hline Y_g & Z_g \end{array}\right]$$

  for some matrices X_g, Y_g, Z_g where X_g is a $k \times k$ matrix.
- Then, if we define $\rho(g) = X_g$ and $\phi(g) = Z_g$, both ρ and ϕ are representations of G.

Reducibility of FG-Modules

An nonzero FG-module is called *reducible* if it has an FG-submodule not equal to $\{0\}$ or V. Otherwise, it is *irreducible*.

- This is the same process we used to decompose the $\mathbb{C}D_8$-module in the previous example:

$$\underset{id}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \quad \underset{r}{\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}} \quad \underset{r^2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \quad \underset{r^3}{\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}}$$

$$\underset{s}{\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}} \quad \underset{sr}{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}} \quad \underset{sr^2}{\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}} \quad \underset{sr^3}{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}$$

- Because the top-right entries were all 0, we were able to obtain the red and blue representations.

Direct Sums

We can also combine two FG-modules: if V and W are two FG-modules, then the space $V \oplus W$ also forms an FG-module if we define multiplication as follows:

- For any vector $x \in V \oplus W$, x can be written as $v + w$ for unique vectors $v \in V$ and $w \in W$. Then, define

$$gx = gv + gw$$

  for all $g \in G$.

Direct Sums

- For example, if ρ and ϕ are representations corresponding to the $\mathbb{C}D_8$-modules V and W, then we can obtain the representation $\rho \oplus \phi$ corresponding to the $\mathbb{C}D_8$-module $V \oplus W$ in the following manner:

$$\rho = \begin{matrix} id & r & r^2 & r^3 \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \\ s & sr & sr^2 & sr^3 \\ \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \end{matrix}$$

$$\phi = \begin{matrix} id & r & r^2 & r^3 & s & sr & sr^2 & sr^3 \\ [1] & [1] & [1] & [1] & [1] & [1] & [1] & [1] \end{matrix}$$

Direct Sums

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$$\rho \oplus \phi = \begin{matrix} id & r & r^2 & r^3 \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & -2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ s & sr & sr^2 & sr^3 \\ \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

Group Algebra

The *Group Algebra* FG is the $|G|$ dimensional vector space of all expressions of the form $\sum_{g \in G} \lambda_g g$ with multiplication given by

$$(\sum_{g \in G} \lambda_g g)(\sum_{h \in G} \mu_h h) = \sum_{g, h \in G} \lambda_g \mu_h (gh)$$

It is referred to as the *regular FG*-module and the representation which arises from this is the *regular* representation.

Theorem

Write the regular $\mathbb{C}G$-module as the direct sum of irreducible $\mathbb{C}G$-modules as

$$\mathbb{C}G = U_1 \oplus U_2 \oplus \ldots \oplus U_r$$

Then every irreducible $\mathbb{C}G$-module is isomorphic to some U_i.

Group Algebra of D_8

$$D_8 = \left\langle r, s \ : \ r^4 = 1, s^2 = 1, srs^{-1} = r^{-1} \right\rangle$$

Let $u, v \in \mathbb{C}D_8$ such that

$$u = 3 + r^3 + 2s \quad v = 6r + 5sr$$

Then

$$\begin{aligned} uv &= (3 + r^3 + 2s)(6r + 5sr) \\ &= 18(r) + 15(sr) + 6(r^3)(r) + 5(r^3)(sr) + 12(s)(r) + 10(s)(sr) \\ &= 18r + 15sr + 6 + 5sr^2 + 12sr + 10r \\ &= 6 + 28r + 27sr + 5sr^2 \end{aligned}$$

Maschke's Theorem

Theorem

Let F be $\mathbb{R}$ or $\mathbb{C}$, and let V be an FG-module. If U is an FG-submodule of V, then there is an FG-submodule W of V such that $V = U \oplus W$.

Corollary

V is completely reducible if it can be written as $V = U_1 \oplus U_2 \oplus \ldots \oplus U_r$, where each U_i is an irreducible FG-module. If F is $\mathbb{C}$ or $\mathbb{R}$, then every non-zero FG-module is completely reducible.

Decomposition of Group Algebra

Take $V = \mathbb{C}D_8$, and $U \subset V$ such that

$$U = span\left(\sum_{g \in G} g\right) = span(1 + r + r^2 + \ldots + sr^3)$$

.

We want to find W such that $V = U \oplus W$.
Taking

$$W = \left\{\sum_{g \in G} a_g g : \sum_{g \in G} a_g = 0\right\}$$

gives two G-invariant subspaces whose direct sum is clearly V.

Schur's Lemma

Let V and W be irreducible $\mathbb{C}G$-modules.

1. If $\phi : V \to W$ is a $\mathbb{C}G$-homomorphism, then either ϕ is a $\mathbb{C}G$-isomophism or $\phi(v) = 0$ for all $v \in V$.
2. If $\phi : V \to V$ is a $\mathbb{C}G$-homomorphism, then ϕ is a scalar multiple of the identity endomorphism 1_V.

Corollary

If every irreducible $\mathbb{C}G$-module of a finite group G has dimension one, then G is abelian.

Using this fact we can prove that every group of order $|G| = p^2$ is abelian.

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