Character Theory of Finite Groups

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Motivation

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- Representation theory gives us a nice way of translating abstract relations into an easier language.
- We will focus on the finite representation of groups and work with vector spaces over $\mathbb{C}$. We pick $\mathbb{C}$ because it is algebraically closed and has characteristic 0.

Basic Definitions

Definition

A representation of a group G is the pair (V, ρ) where V is a vector space and ρ is a group homomorphism from $G \to \mathrm{GL}(V)$, i.e. $\rho(g_1)\rho(g_2) = \rho(g_1 g_2)$.

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Definition

Given a group G and representations V and W, let $\mathrm{Hom}_G(V, W)$ be the linear maps $\phi : V \to W$ with $\phi\rho_V(g) = \rho_W(g)\phi$.

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Lemma (Schur)

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Theorem (Maschke)

Let V be any representation of G. Then V is the direct sum of simple representations of G.

Examples of Representations

Example (C_3)

The regular representation of C_3 is $\mathbb{C}^3$ where the action of $g \in C_3$ is cyclically permuting the coordinates.

- The space (a, a, a) is the trivial representation.
- The space $(a, b, c) : a + b + c = 0$ is a two-dimensional subrepresentation.

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Example (S_3)

We give examples of irreducible representations of S_3.

- The trivial representation, $\mathbb{C}_+$, which sends all g to 1.
- The sign representation, $\mathbb{C}_-$, which sends all elements to $\operatorname{sgn}(g) \in \{-1, +1\}$.
- The space $(a, b, c) : a + b + c = 0$, $\mathbb{C}^2$, where g acts by permutation of coordinates.

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Lemma

If V and W are representations of G, then $\chi_{V\oplus W} = \chi_V + \chi_W$.

$$\chi_{V\oplus W}(g) = \text{Tr}\begin{bmatrix} \rho_V(g) & 0 \\ 0 & \rho_W(g) \end{bmatrix} = \text{Tr}(\rho_V(g)) + \text{Tr}(\rho_W(g))$$

$$= \chi_V(g) + \chi_W(g)$$

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- The trivial representation, $\mathbb{C}_+$, has character $\chi(g) = 1$.
- The sign representation, $\mathbb{C}_-$, has character $\chi(g) = \mathrm{sgn}(g)$.
- The space $(a, b, c) : a + b + c = 0$ where g acts by permutation of coordinates is the mean zero representation, $\mathbb{C}^2$. Thus, $\chi(g)$ is one less than the number of fixed points of g.

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- What kind of structure do characters have?
- It can be shown from Maschke's Theorem that characters of simple representations are linearly independent and span the vector space $F_c(G, \mathbb{C})$ of class functions $G \to \mathbb{C}$.
- Define an inner product $(-,-)$ on $F_c(G, \mathbb{C})$ by

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

or, letting $\{C_i\}$ be the conjugacy classes of G,

$$\sum_i \frac{|C_i|}{|G|} f_1(C_i) \overline{f_2(C_i)}.$$

Orthogonality Relations

Theorem (Orthogonality by rows)

For V, W simple, $(\chi_V, \chi_W) = \begin{cases} 1 & V \cong W \\ 0 & V \ncong W \end{cases}$

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- Proof sketch: it can be shown that

$$\frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_W(g)} = \dim \mathrm{Hom}_G(W, V).$$

The result then follows immediately from Schur's lemma.

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- Thus this basis is orthonormal with respect to $(-, -)$.

Orthogonality Relations, Cont.

- A different orthonormal basis is given by $\{\sqrt{|G|/|C_i|}\delta_i\}$, where
  $\delta_i(g) = \begin{cases} 1 & g \in C_i \\ 0 & g \notin C_i \end{cases}$

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- Some calculation gives $(\delta_i, \delta_j) = \sum_V \chi_V(C_i)\chi_V(C_j)$, where the sum is over simple representations. This leads to

Theorem (Orthogonality by columns)

$$\sum_V \chi_V(C_i)\chi_V(C_j) = \begin{cases} |G|/|C_i| & i = j \\ 0 & i \neq j \end{cases}$$

Character Tables

- These data can be summarized in a character table. Rows are indexed by simples, columns by conjugacy classes. The number in row V and column C is $\chi_V(C)$. A row giving the size of each conjugacy class is also included.

Example (S_3)

S_3	1^3	$1^1 2^1$	3^1
#	1	3	2
$\mathbb{C}_+$	1	1	1
$\mathbb{C}_-$	1	-1	1
$\mathbb{C}^2$	2	0	-1

Conclusion

- This information, together with Schur's lemma and Maschke's theorem, can be used to extract the simple summands (with multiplicity) of any representation of G, which determine it up to isomorphism.

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Conclusion

- This information, together with Schur's lemma and Maschke's theorem, can be used to extract the simple summands (with multiplicity) of any representation of G, which determine it up to isomorphism.
- Furthermore, this is accomplished with a easy, concrete computation. Operations such as taking quotients and tensor products are similarly tractable with this machine.
- Thus the character table of a finite group gives an essentially complete description of its representation theory as well as a powerful computational tool for working with ostensibly abstract objects.

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