The Rational Cherednik Algebra of Type A_1 with Divided Powers in Characteristic p

Lev Kruglyak
Mentor: Daniil Kalinov

University High School, Irvine

May 18-19, 2019
MIT PRIMES Conference

Basic Definitions

The **characteristic** of a field is smallest number of 1's which add up to 0. If no such number exists, we say the field has characteristic 0.

Basic Definitions

The **characteristic** of a field is smallest number of 1's which add up to 0. If no such number exists, we say the field has characteristic 0.

1. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields of characteristic 0.

Basic Definitions

The **characteristic** of a field is smallest number of 1's which add up to 0. If no such number exists, we say the field has characteristic 0.

1. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields of characteristic 0.
2. $\mathbb{F}_p$ is a field of characteristic p.

Basic Definitions

The **characteristic** of a field is smallest number of 1's which add up to 0. If no such number exists, we say the field has characteristic 0.

1. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields of characteristic 0.
2. $\mathbb{F}_p$ is a field of characteristic p.

An **algebra** is a k-vector space V with a bilinear multiplication $V \times V \to V$.

Basic Definitions

The **characteristic** of a field is smallest number of 1's which add up to 0. If no such number exists, we say the field has characteristic 0.

1. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields of characteristic 0.
2. $\mathbb{F}_p$ is a field of characteristic p.

An **algebra** is a k-vector space V with a bilinear multiplication $V \times V \to V$.

Examples of algebras:

1. k — the base field as an algebra over itself

Basic Definitions

The **characteristic** of a field is smallest number of 1's which add up to 0. If no such number exists, we say the field has characteristic 0.

1. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields of characteristic 0.
2. $\mathbb{F}_p$ is a field of characteristic p.

An **algebra** is a k-vector space V with a bilinear multiplication $V \times V \to V$.

Examples of algebras:

1. k — the base field as an algebra over itself
2. $k[x_1, \cdots, x_n]$ — polynomials in n variables under multiplication

Basic Definitions

The **characteristic** of a field is smallest number of 1's which add up to 0. If no such number exists, we say the field has characteristic 0.

1. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields of characteristic 0.
2. $\mathbb{F}_p$ is a field of characteristic p.

An **algebra** is a k-vector space V with a bilinear multiplication $V \times V \to V$.

Examples of algebras:

1. k — the base field as an algebra over itself
2. $k[x_1, \cdots, x_n]$ — polynomials in n variables under multiplication
3. $k[x_1, \cdots, x_n, \partial_1, \cdots, \partial_n]$ — differential operators under composition

Basic Definitions

The **characteristic** of a field is smallest number of 1's which add up to 0. If no such number exists, we say the field has characteristic 0.

1. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields of characteristic 0.
2. $\mathbb{F}_p$ is a field of characteristic p.

An **algebra** is a k-vector space V with a bilinear multiplication $V \times V \to V$.

Examples of algebras:

1. k — the base field as an algebra over itself
2. $k[x_1, \cdots, x_n]$ — polynomials in n variables under multiplication
3. $k[x_1, \cdots, x_n, \partial_1, \cdots, \partial_n]$ — differential operators under composition
4. $\mathrm{End}(V)$ — endomorpisms of a vector space
    1. consists of linear maps $V \to V$ under composition

Basic Definitions

The **characteristic** of a field is smallest number of 1's which add up to 0. If no such number exists, we say the field has characteristic 0.

1. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields of characteristic 0.
2. $\mathbb{F}_p$ is a field of characteristic p.

An **algebra** is a k-vector space V with a bilinear multiplication $V \times V \to V$.

Examples of algebras:

1. k — the base field as an algebra over itself
2. $k[x_1, \cdots, x_n]$ — polynomials in n variables under multiplication
3. $k[x_1, \cdots, x_n, \partial_1, \cdots, \partial_n]$ — differential operators under composition
4. $\mathrm{End}(V)$ — endomorpisms of a vector space
    1. consists of linear maps $V \to V$ under composition

A **subalgebra** $E \subset A$ is a subset of an algebra which is closed under the operations. For example, $k[x^2] \subset k[x]$.

Divided Power Extensions

Goal: To extend an algebra of polynomial operators $A \subset \mathrm{End}(k[x])$ from characteristic 0 to characteristic p in a nice way.

Divided Power Extensions

Goal: To extend an algebra of polynomial operators $A \subset \mathrm{End}(k[x])$ from characteristic 0 to characteristic p in a nice way.

1. One way would be to reduce the algebra mod p.

Divided Power Extensions

Goal: To extend an algebra of polynomial operators $A \subset \mathrm{End}(k[x])$ from characteristic 0 to characteristic p in a nice way.

1. One way would be to reduce the algebra mod p.
    1. This leads to a problem: For example $\partial^3 x^n = n(n-1)(n-2)x^{n-3}$ is always zero mod 3. Many of the differential operators become zero when reducing mod p.

Divided Power Extensions

Goal: To extend an algebra of polynomial operators $A \subset \text{End}(k[x])$ from characteristic 0 to characteristic p in a nice way.

1. One way would be to reduce the algebra mod p.
   1. This leads to a problem: For example $\partial^3 x^n = n(n-1)(n-2)x^{n-3}$ is always zero mod 3. Many of the differential operators become zero when reducing mod p.
2. Instead, we can divide operators maximally by powers of p and then reduce mod p so we don't 'lose' operators.

Divided Power Extensions

Goal: To extend an algebra of polynomial operators $A \subset \mathrm{End}(k[x])$ from characteristic 0 to characteristic p in a nice way.

1. One way would be to reduce the algebra mod p.
   1. This leads to a problem: For example $\partial^3 x^n = n(n-1)(n-2)x^{n-3}$ is always zero mod 3. Many of the differential operators become zero when reducing mod p.
2. Instead, we can divide operators maximally by powers of p and then reduce mod p so we don't 'lose' operators.
   For example, $\frac{\partial^3}{3} x^n = 2\binom{n}{3} x^{n-3}$, this is nonzero mod 3.

Divided Power Extensions

Goal: To extend an algebra of polynomial operators $A \subset \text{End}(k[x])$ from characteristic 0 to characteristic p in a nice way.

1. One way would be to reduce the algebra mod p.
   1. This leads to a problem: For example $\partial^3 x^n = n(n-1)(n-2)x^{n-3}$ is always zero mod 3. Many of the differential operators become zero when reducing mod p.
2. Instead, we can divide operators maximally by powers of p and then reduce mod p so we don't 'lose' operators.
   For example, $\frac{\partial^3}{3} x^n = 2\binom{n}{3} x^{n-3}$, this is nonzero mod 3.
   1. After dividing every operator maximally by p, we are left with a **divided power extension** of A.

Example: Algebra of Differential Operators

The algebra of polynomial differential operators over a field, denoted $k[x, \partial]$, is generated by x and ∂, where x acts by multiplication and ∂ acts by differentiation.

Example: Algebra of Differential Operators

The algebra of polynomial differential operators over a field, denoted $k[x, \partial]$, is generated by x and ∂, where x acts by multiplication and ∂ acts by differentiation.

Define the k-th **Hasse derivative** as $\partial^{(k)} x^n = \frac{\partial^k}{k!} x^n = \binom{n}{k} x^{n-k}$. These are 'divided power' derivatives in characteristic p.

Example: Algebra of Differential Operators

The algebra of polynomial differential operators over a field, denoted $k[x, \partial]$, is generated by x and ∂, where x acts by multiplication and ∂ acts by differentiation.

Define the k-th **Hasse derivative** as $\partial^{(k)} x^n = \frac{\partial^k}{k!} x^n = \binom{n}{k} x^{n-k}$. These are 'divided power' derivatives in characteristic p.

In fact, given any differential operator, after dividing maximally by p and reducing mod p, it can be written in terms of Hasse derivatives.

Example: Algebra of Differential Operators

The algebra of polynomial differential operators over a field, denoted $k[x, \partial]$, is generated by x and ∂, where x acts by multiplication and ∂ acts by differentiation.

Define the k-th **Hasse derivative** as $\partial^{(k)} x^n = \frac{\partial^k}{k!} x^n = \binom{n}{k} x^{n-k}$. These are 'divided power' derivatives in characteristic p.

In fact, given any differential operator, after dividing maximally by p and reducing mod p, it can be written in terms of Hasse derivatives.

It suffices to use only prime power Hasse derivatives, i.e. $\partial^{(1)}, \partial^{(p)}, \partial^{(p^2)}, \ldots$ These operators generate the divided power extension of the algebra of differential operators.

Example: Algebra of Differential Operators

The algebra of polynomial differential operators over a field, denoted $k[x, \partial]$, is generated by x and ∂, where x acts by multiplication and ∂ acts by differentiation.

Define the k-th **Hasse derivative** as $\partial^{(k)} x^n = \frac{\partial^k}{k!} x^n = \binom{n}{k} x^{n-k}$. These are 'divided power' derivatives in characteristic p.

In fact, given any differential operator, after dividing maximally by p and reducing mod p, it can be written in terms of Hasse derivatives.

It suffices to use only prime power Hasse derivatives, i.e. $\partial^{(1)}, \partial^{(p)}, \partial^{(p^2)}, \ldots$ These operators generate the divided power extension of the algebra of differential operators.

In our project, we want to calculate the divided power extension of a Cherednik algebra. These algebras have many applications in mathematical physics and are of interest to representation theorists.

Cherednik Algebra

Consider the **reflection** operator s which acts on x^n as $(-1)^n x^n$.

Cherednik Algebra

Consider the **reflection** operator s which acts on x^n as $(-1)^n x^n$.

For some complex number c, define the **Dunkl operator** as $D = \partial + \frac{c}{2x}(1 - s)$.

Cherednik Algebra

Consider the **reflection** operator s which acts on x^n as $(-1)^n x^n$.

For some complex number c, define the **Dunkl operator** as $D = \partial + \frac{c}{2x}(1 - s)$.
For example, $D^3 x^{2n} = 2n(2n - 1 + c)(2n - 2)x^{2n-3}$.

Cherednik Algebra

Consider the **reflection** operator s which acts on x^n as $(-1)^n x^n$.

For some complex number c, define the **Dunkl operator** as $D = \partial + \frac{c}{2x}(1 - s)$. For example, $D^3 x^{2n} = 2n(2n - 1 + c)(2n - 2)x^{2n-3}$.

Definition

The **rational Cherednik algebra of type** A_1, denoted by H_c, is the algebra generated by the operators x, s and D.

Cherednik Algebra

Consider the **reflection** operator s which acts on x^n as $(-1)^n x^n$.

For some complex number c, define the **Dunkl operator** as $D = \partial + \frac{c}{2x}(1-s)$. For example, $D^3 x^{2n} = 2n(2n-1+c)(2n-2)x^{2n-3}$.

Definition

The **rational Cherednik algebra of type** A_1, denoted by H_c, is the algebra generated by the operators x, s and D.

In this presentation, I will only discuss our results in the spherical subalgebra, a 'reduced' form of the Cherednik algebra.

Cherednik Algebra

Consider the **reflection** operator s which acts on x^n as $(-1)^n x^n$.

For some complex number c, define the **Dunkl operator** as $D = \partial + \frac{c}{2x}(1-s)$. For example, $D^3 x^{2n} = 2n(2n-1+c)(2n-2)x^{2n-3}$.

Definition

The **rational Cherednik algebra of type** A_1, denoted by H_c, is the algebra generated by the operators x, s and D.

In this presentation, I will only discuss our results in the spherical subalgebra, a 'reduced' form of the Cherednik algebra.

Definition

Consider the symmetrizer, $\mathbf{e} = \frac{1+s}{2}$. This annihilates all odd degree terms, and fixes even degree terms. The **spherical subalgebra** is defined as $B_c = \mathbf{e}H_c\mathbf{e}$.

Cherednik Algebra

Consider the **reflection** operator s which acts on x^n as $(-1)^n x^n$.

For some complex number c, define the **Dunkl operator** as $D = \partial + \frac{c}{2x}(1-s)$. For example, $D^3 x^{2n} = 2n(2n-1+c)(2n-2)x^{2n-3}$.

Definition

The **rational Cherednik algebra of type** A_1, denoted by H_c, is the algebra generated by the operators x, s and D.

In this presentation, I will only discuss our results in the spherical subalgebra, a 'reduced' form of the Cherednik algebra.

Definition

Consider the symmetrizer, $\mathbf{e} = \frac{1+s}{2}$. This annihilates all odd degree terms, and fixes even degree terms. The **spherical subalgebra** is defined as $B_c = \mathbf{e}H_c\mathbf{e}$.

So the spherical subalgebra consists of even degree operators acting on even degree monomials, and no s terms.

p-adic Valuation Formula for Dunkl Operators

The most interesting case is when c is an integer. Let c_i be the remainder when c is divided by p^i. Define $d_i(c)$ as,

$$d_i(c) = \begin{cases} p^i & \text{if } c_i = 0 \\ p^i - c_i + 1 & \text{if } c_i \text{ is even} \\ c_i & \text{if } c_i \text{ is odd} \end{cases}.$$

p-adic Valuation Formula for Dunkl Operators

The most interesting case is when c is an integer. Let c_i be the remainder when c is divided by p^i. Define $d_i(c)$ as,

$$d_i(c) = \begin{cases} p^i & \text{if } c_i = 0 \\ p^i - c_i + 1 & \text{if } c_i \text{ is even} \\ c_i & \text{if } c_i \text{ is odd} \end{cases}.$$

We proved a p-adic valuation formula for the Dunkl operator D^k,

$$\nu_p(D^k) = \sum_{i=1}^{\infty} \left(\left\lfloor \frac{k}{2p^i} \right\rfloor + \left\lfloor \frac{k + d_i(c)}{2p^i} \right\rfloor \right).$$

Divided Powers in the Spherical Subalgebra

Similarly to the algebra of differential operators, we want to figure out a basis of divided powers.

Divided Powers in the Spherical Subalgebra

Similarly to the algebra of differential operators, we want to figure out a basis of divided powers.

Let S_c be the minimal set of powers of the Dunkl operator needed to express divided powers of an arbitrary power of a Dunkl operator.

Divided Powers in the Spherical Subalgebra

Similarly to the algebra of differential operators, we want to figure out a basis of divided powers.

Let S_c be the minimal set of powers of the Dunkl operator needed to express divided powers of an arbitrary power of a Dunkl operator.

For example, in the algebra of differential operators, we only needed prime powers of derivatives.

Divided Powers in the Spherical Subalgebra

Similarly to the algebra of differential operators, we want to figure out a basis of divided powers.

Let S_c be the minimal set of powers of the Dunkl operator needed to express divided powers of an arbitrary power of a Dunkl operator.

For example, in the algebra of differential operators, we only needed prime powers of derivatives.

We explicitly constructed S_c, and noticed a really interesting fractal pattern emerging. In the next slide, each vertical slice represents S_c where c is the x axis.

Visual Representation of the Divided Powers

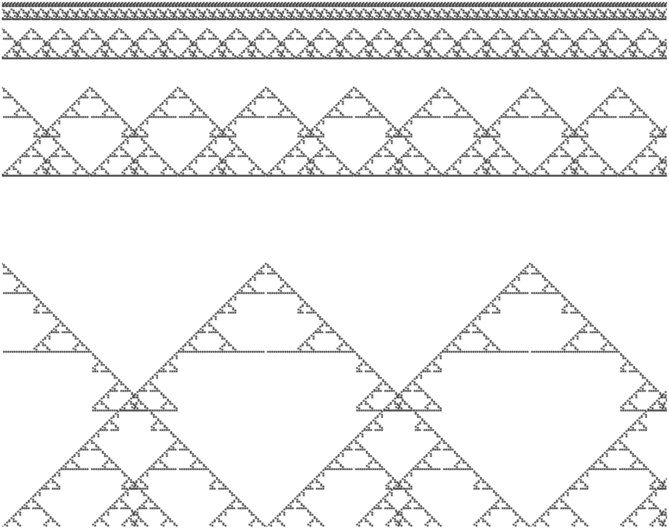

Figure: The sets S_c when $p = 3$

Visual Representation of the Divided Powers

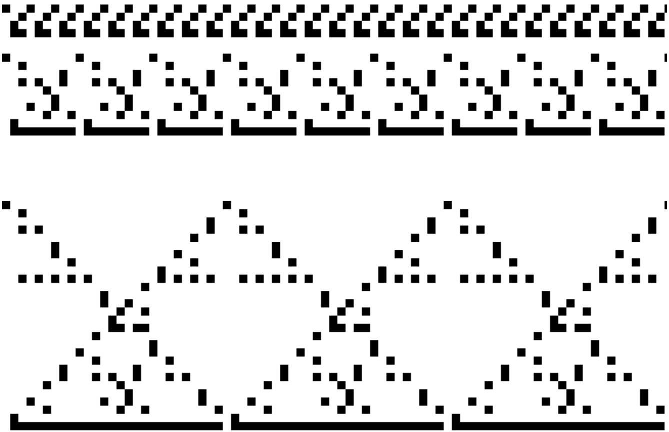

Figure: The sets S_c when $p = 3$ (zoomed in version)

For example, $S_{29} = \{4, 6, 10, 12, 18, 28, 30, 36, 54, 133, \ldots\}$

Grothendieck-style Definition

In our research, we also provided a more abstract definition of the divided power extension and proved its equivalence with the combinatorial definition.

Grothendieck-style Definition

In our research, we also provided a more abstract definition of the divided power extension and proved its equivalence with the combinatorial definition.

Definition

Let $D(k)$ be the algebra of rational polynomial differential operators over k, including Hasse derivatives in characteristic p. For any $a \in k$, define the **stabilizer** of a as,

$$S_a(k) = \{Q \in D(k) : x^{-a} Q x^a (k[x^2]) \subset k[x^2]\}.$$

Grothendieck-style Definition

In our research, we also provided a more abstract definition of the divided power extension and proved its equivalence with the combinatorial definition.

Definition

Let $D(k)$ be the algebra of rational polynomial differential operators over k, including Hasse derivatives in characteristic p. For any $a \in k$, define the **stabilizer** of a as,

$$S_a(k) = \{Q \in D(k) : x^{-a} Q x^a (k[x^2]) \subset k[x^2]\}.$$

Define the **Grothendieck spherical subalgebra** as $\mathcal{B}_c(k) = S_0 \cap S_{1-c}$

Grothendieck-style Definition

In our research, we also provided a more abstract definition of the divided power extension and proved its equivalence with the combinatorial definition.

Definition

Let $D(k)$ be the algebra of rational polynomial differential operators over k, including Hasse derivatives in characteristic p. For any $a \in k$, define the **stabilizer** of a as,

$$S_a(k) = \{Q \in D(k) : x^{-a} Q x^a (k[x^2]) \subset k[x^2]\}.$$

Define the **Grothendieck spherical subalgebra** as $\mathcal{B}_c(k) = S_0 \cap S_{1-c}$

Theorem

1. *In characteristic* 0, *we have* $\mathcal{B}_c(k) = B_c$

Grothendieck-style Definition

In our research, we also provided a more abstract definition of the divided power extension and proved its equivalence with the combinatorial definition.

Definition

Let $D(k)$ be the algebra of rational polynomial differential operators over k, including Hasse derivatives in characteristic p. For any $a \in k$, define the **stabilizer** of a as,

$$S_a(k) = \{Q \in D(k) : x^{-a} Q x^a (k[x^2]) \subset k[x^2]\}.$$

Define the **Grothendieck spherical subalgebra** as $\mathcal{B}_c(k) = S_0 \cap S_{1-c}$

Theorem

1. *In characteristic* 0, *we have* $\mathcal{B}_c(k) = B_c$
2. *In characteristic* p, *we have* $\mathcal{B}_c(k) = \mathcal{DP}_{B_c}$, *where* $\mathcal{DP}_{B_c}$ *is the divided power extension of the spherical subalgebra. (Still a conjecture)*

Future/Current Research

1. So far, we've only created a basis of divided powers for single term operators, we have not finished with sums of Dunkl operators.

Future/Current Research

1. So far, we've only created a basis of divided powers for single term operators, we have not finished with sums of Dunkl operators.
2. We are currently working in type A_1, where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{C}$. This is where the reflection operators come from. In the future, we would like to work on type A_n, where S^n acts on $\mathbb{C}^{n-1}$. These algebras are more complicated, generated by Dunkl operators and group elements.

Acknowledgements

I would like to sincerely thank:

1. My parents, for supporting me
2. My mentor, Daniil Kalinov for meeting with me twice a week and giving me valuable feedback on all my background reading problems and research
3. Dr. Tanya Khovanova, for meeting with me several times in the last few months and giving me advice on the paper and presentation
4. Prof. Pavel Etingof, for proposing and overseeing this project
5. Dr. Slava Gerovitch and the MIT PRIMES Program, for giving me this amazing opportunity