Mixed Strategy Equilibria for the Five Front Winner Takes All Variant of Colonel Blotto

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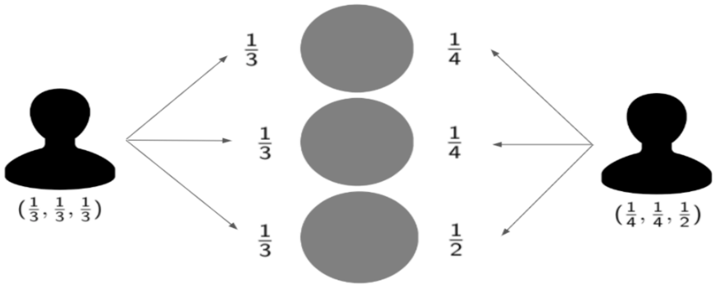

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The strategy that plays...

1. $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ with probability $\frac{1}{2}$
2. $(\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$ with probability $\frac{1}{3}$
3. $(\frac{1}{5}, \frac{1}{5}, \frac{3}{5})$ with probability $\frac{1}{6}$

is a mixed strategy.

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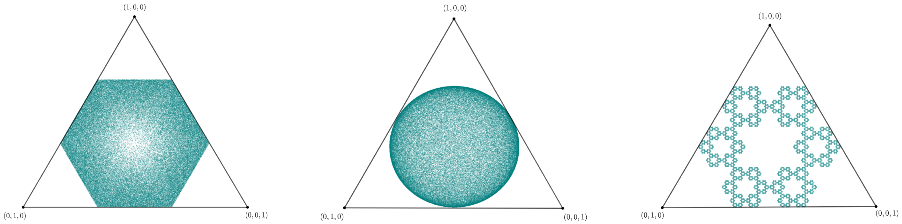
A Measure On The Triangle of Pure Strategies



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Various discretizations of the General Colonel Blotto have been studied:

1. Numerous mixed equilibrium existence results have been proven.
2. Many such results focus on strategies that maximize the **expected number** of fronts (or weight) won, rather than probability of winning the **majority** of the fronts (or weight).

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Defining Property of Continuous Equilibrium Mixed Strategy

Let S be the (infinite) set of pure strategies. An equilibrium mixed strategy is a probability distribution μ on S such that for all $s' \in S$

$$\int_S p(s, s')\mathrm{d}\mu(s) \geq 0.5.$$

Where $p(s, s')$ is the probability that a random permutation of s wins against s'.

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Any mixed strategy that maximizes expectation can be shown to expect to lose at least half the time to the pure strategy $(0, 0, \frac{3}{10}, \frac{3}{10}, \frac{2}{5})$.

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Theorem (Brady, Erives)

There exists a mixed strategy equilibrium for $2K + 1$ Front Winner Takes All Colonel Blotto, for all $K \geq 1$.

Formulation of Discretization as a LP

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This is a set of $|S|$ inequalities in $|S|$ variables and can be viewed as a linear program, and solved as such by a computer.

Discretization Small Example – $N = 5$

A equilibrium mixed strategy for $N = 5$:

1. $[1, 1, 1, 1, 1]$ with probability 0.3333
2. $[0, 1, 1, 1, 2]$ with probability 0.3333
3. $[0, 0, 1, 2, 2]$ with probability 0.3333
4. $[0, 0, 1, 1, 3]$ with probability 0.0
5. $[0, 0, 0, 2, 3]$ with probability 0.0
6. $[0, 0, 0, 1, 4]$ with probability 0.0
7. $[0, 0, 0, 0, 5]$ with probability 0.0

Discretization Larger Example – $N = 45$

Seven most played pure strategies in a mixed equilibrium strategy for $N = 45$:

1. $[2, 3, 10, 14, 16]$ with probability 0.0433
2. $[0, 0, 13, 14, 18]$ with probability 0.0415
3. $[1, 6, 8, 14, 16]$ with probability 0.0381
4. $[0, 0, 11, 16, 18]$ with probability 0.0359
5. $[0, 8, 10, 10, 17]$ with probability 0.0333
6. $[1, 6, 10, 10, 18]$ with probability 0.0327
7. $[2, 4, 9, 12, 18]$ with probability 0.0324
8. $\dots$

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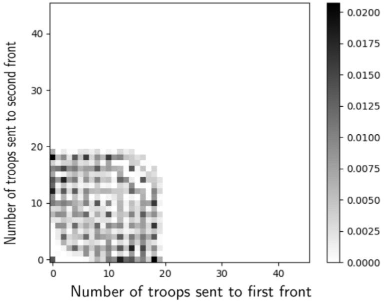
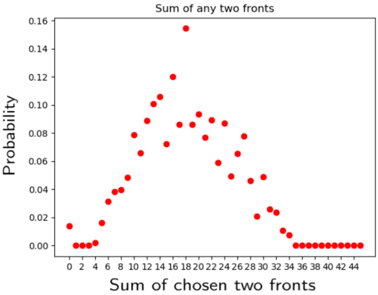
Conjecture

For all pure strategies s, if $T(s) > 0$, then s sends at most $\frac{2N}{5} + 1$ troops to a single front, where N is the discretization parameter.

Visualizations for the $N = 45$ discretization



Distribution over pairs (x, y) sent to a pair of fronts.



Distribution over amount of units of army sent to two fronts.

Future work

1. Characterize equilibrium strategies for Five-Front Winner Takes All Colonel Blotto
2. Extend these results to $K > 5$ fronts.

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5. My parents