Weighing Coins, Losing Weight, and Saving Money

Isha Agarwal, Paul Braverman, Patrick Chen, William Du, Kaylee Ji, Akhil Kammila, Shane Lee, Alicia Li, Anish Mudide, Jeffrey Shi, Maya Smith, Isabel Tu

Mentor: Tanya Khovanova

MIT PRIMES STEP Junior Group

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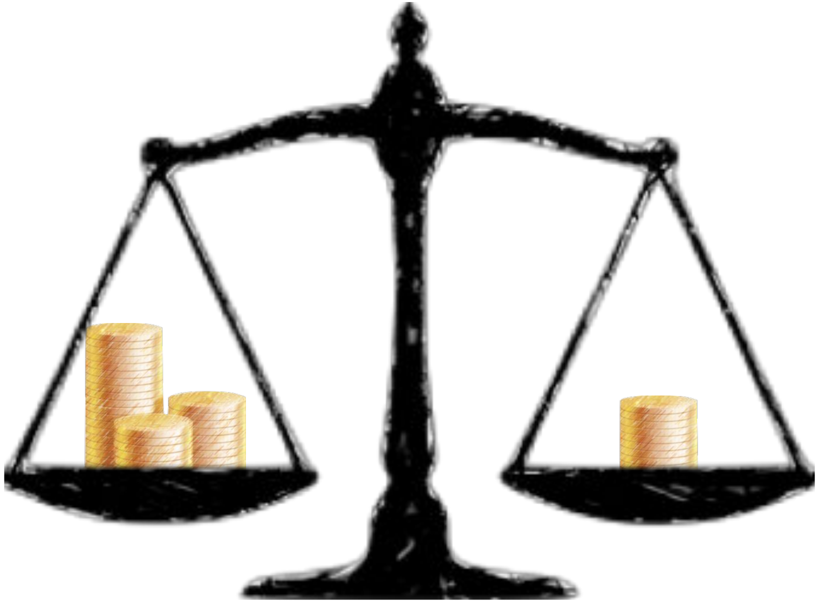
The Original Problem

You have six bags with coins that look the same. Each bag has an infinite number of coins and all coins in the same bag weigh the same amount. Coins in different bags weigh 1, 2, 3, 4, 5 and 6 grams exactly. There is a label (1, 2, 3, 4, 5, 6) attached to each bag that is supposed to correspond to the weight of the coins in that bag. You only have one balance scale. What is the least number of times that you need to weigh coins in order to confirm that the labels are correct?

Source: 2000 Streamline Olympiad. 8th grade.

Answer: One Weighing

70 x 1



1 x 2

2 x 3

3 x 4

4 x 5

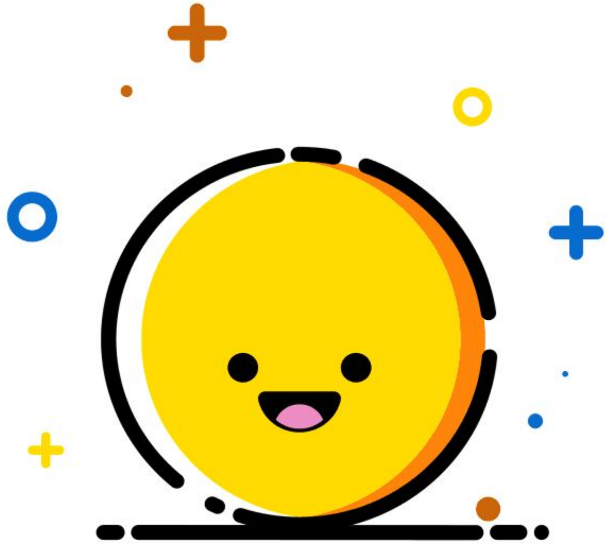
5 x 6

Our *Coin*tinuation

- Let *n* be the number of bags of coins
- We verify if labels are correct in *one weighing*

Our *Coin*ditions and Goals:

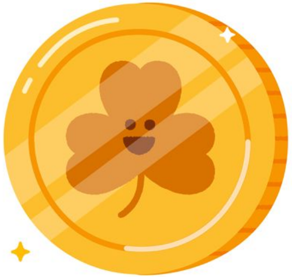
- Minimize total weight used on balance
- Minimize the number of coins
- Having *coin*siderable fun!



Coinfirming the Labels

Verifying Weighing

a weighing that proves all labels on the bags are correct



Getting More Coincise

Coin-optimal - a verifying weighing that minimizes *coins* used

Weight-optimal - a verifying weighing that minimizes weight *on scale*

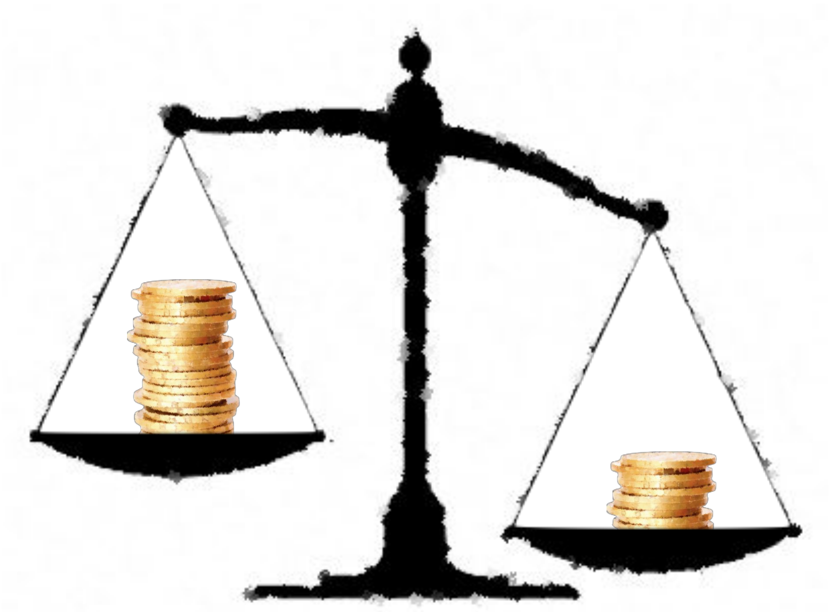
*we sometimes call a weighing **optimal** if it's coin-optimal or weight-optimal

Optimal Solution Coinfiguration

4 x 1

3 x 2

2 x 3



1 x 5

2 x 6

Total Weight: 33

Coins: 12

Notation Coinsistency

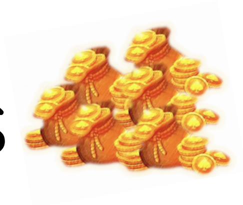
We write 111122233 vs 566 as....

$$\{4,\ 3,\ 2,\ 0,\ -1,\ -2\}$$

...where each number separated by a comma is called a ***multiplicity***

In $\{a_1, a_2, a_3, \ldots\, a_{n-1}, a_n\}$...

- $|a_i|$ is the number of coins of weight i used
- Positive multiplicities represent coins on the left hand side
- Negative multiplicities represent coins on the right hand side
- 0 indicates the coin is not used

Coinsider these Weighings  

# of bags	Multiplicities	Weighing	# of coins	Total Weight
2	{1, -1}	$1 < 2$	2	3
3	{2, 0, -1} OR {2, -1, 0}	$1 + 1 < 3$ OR $1 + 1 = 2$	3	5 OR 4
4	{2, 1, 0, -1}	$1 + 1 + 2 = 4$	4	8
5	{3, 2, 1, 0, -2}	$1 + 1 + 1 + 2 + 2 + 3 = 5 + 5$	8	20
6	{4, 3, 2, 0, -1, -2}	$1 + 1 + 1 + 1 + 2 + 2 + 2 + 3 + 3 < 5 + 6 + 6$	12	33

Rolling downhill...

When a weighing has the following qualities...

- Multiplicities are strictly decreasing
- Lighter coins are on one side (left)
- Heavier coins on other side (right)
- Left sides weighs no more than right side

....we call this a **downhill weighing**

Tight downhill weighing: We proved the difference between the two sides is at most one; ***always verifying***

Minimizing Weight

Separation Point

Separation Point:

- smallest label on the right side of the balance in a downhill weighing
- represented by variable *s*

Example:

- {4,3,2,0,-1,-2}
- Separation point is 5

Minimum Separation Point:

- the separation point that would minimize the total weight
- represented by variable *m*
- Approximately $(2n+4)/3$

Bounding Weights

Bounding Left Weight *(W_L):*

- Weight of coins on the left pan of a downhill weighing
- Minimum possible value written as $W_L(s)$ (for separation point *s*)
- $W_L(s) = \binom{s}{3}$

Bounding Right Weight *(W_R):*

- Weight of coins on the right pan of a downhill weighing
- $W_R(s) = \dfrac{(s-n-2)(s-n-1)(s+2n)}{6}$

Bounding Weights Cointinued

Minimum Bounding Weight:

- Our lower bound for the minimum weight
- Notation: W_B

If $W_L(m) < W_R(m)$ ....then $W_B = 2 \cdot W_R(m) - 1$

If $W_L(m) \geq W_R(m)$ ....then $W_B = 2 \cdot W_L(m)$

Minimum Weight:

- The minimum possible weight
- Notation: W_M

A Simpl**3** **k**ase (*Coin*vien**+1**y)

If the number of bags can be written as $3k+1$, where k is a positive integer

Then the weighing....

$$\{2k, 2k-1, ..., 0, -1, -2, ... , -(k-1), -k\}$$

.....is *always* verifying

Separation point: $2k+2$

Minimum Weight: $2 \cdot \binom{2k+2}{3}$ OR $\frac{8n^3 + 12n^2 - 12n - 8}{81}$

Minimum Coins: $\frac{k(5k+3)}{2}$ OR $\frac{(n-1)(5n+4)}{18}$

Another Simpl**3 k**ase

For this case, the minimum separation point is 2k+1

The minimum weight is:

- $W_M(3k) = W_B(3k) = 2 \bullet W_R(2k+3) - 1 =$ $\dfrac{8k^3 + 9k^2 + k}{3} - 1 = \dfrac{8n^3 + 27n^2 + 9n - 81}{81}.$
- Using this, we can see that low minimum weights are 5, 33, 99, 219, and so on

A not so Simpl**3 k**ase (I**+**'s Cra2ier)

The minimal separation point is 2k+3, but $\frac{(3k+2)(k+1)}{2}$ coins must be added to the right side

Subcase n=3k+2, k odd

- The largest weight can be added (k+1)/2 times
- Minimum bounding weight: $\frac{8n^3+56n^2-58n-10}{81}$

Subcase n=3k+2, k even

- The bounding weight is achievable for n>50
- The weights follow a pattern of 20, 70, 168, 330, and so on

Coin Lower Bound

The Lower Bound is a threshold defined as the minimum amount of coins a verifying weighing must have.

It is approximately $\frac{n^2}{4}$ where n represents the number of bags.

If the number of coins in a weighing is less than the Lower Bound, it cannot be verifying.

Coin Upper Bound

The Upper Bound is the maximum amount of coins that is needed to create a verifying weighing.

It is approximately $\frac{2}{3}n^3$ where n represents the number of bags.

If the number of coins in a weighing is more than the Upper Bound, it is possible to create a verifying weighing.

Some Other Coinpelling Cases

Solo weighings are when there is exactly one type of coin on the right side. Some examples for a small *n* are:

Number of Bags	Multiplicities	Number of Coins
n = 4	{2, 1, 0, -1}	4
n = 5	{3, 2, 1, 0, -2} OR {4, 3, 2, 1, -4}	8 OR 14
n = 6	{5, 4, 3, 2, 1, -6} (unbalanced)	21
n = 7	{5, 4, 3, 2, 1, 0, -5} OR {6, 5, 4, 3, 2, 1, -8}	20 OR 29
n = 8	{6, 5, 4, 3, 2, 1, 0, -7}	28
n = 9	{8, 7, 5, 4, 3, 2, 1, 0, -10} (this isn't a downhill weighing)	40

Two Lemmas.

1. A downhill solo weighing such that the left part is consecutive and there is no multiplicity zero exists if and only if the number of bags has remainder 1 or –1 when divided by 6.
2. A downhill solo weighing such that the left part is consecutive and there is multiplicity zero exists if and only if the number of bags is not divisible by 3.

Some Other Coinpelling Cases (Cointinued)

Arithmetic progression weighings are when the multiplicities in a balanced weighing form an arithmetic progression

Primitive weighings are weighings with a set of multiplicities A such that gcd(A) = 1

- A primitive arithmetic progression weighing occurs if and only if:
- the difference **d=3** OR the difference is **d=1 and n=3k+1**

d=3 for a small number of bags - multiplicities are of the general form **2n-2, 2n-5, 2n-8, . . . , 1-n**

Number of bags	Weighing:
3	(4, 1, -2)
4	(6, 3, 0, -3)
5	(8, 5, 2, -1, -4)
6	(10, 7, 4, 1, -2, -5)

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Citations

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