Maximal Extensions of Differential Posets

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Posets

Definition

A *partially ordered set*, or poset, is a set P following the properties:

1. Certain elements $x, y \in P$ are relatable under the binary relation $\leq$.
2. If $x \leq y$ and $y \leq x$ then $x = y$.
3. If $x \leq y$, and $y \leq z$, then $x \leq z$.

Definition

In a poset P, an element y *covers* an element x if $x \leq y$, and there doesn't exist a distinct element z such that $x \leq z \leq y$. We write $x \lessdot y$.

Hasse Diagrams

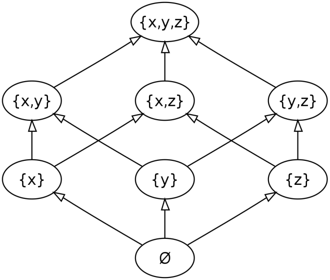


Figure: The Hasse diagram of the set of subsets of (x, y, z)

Posets can be represented in diagrams called *Hasse diagrams*, which appear like directed graphs. An arrow points from the smaller element to the larger element.
In this example, the relation $\leq$ is equivalent to the inclusion relation $\in$.

Example: Young's lattice

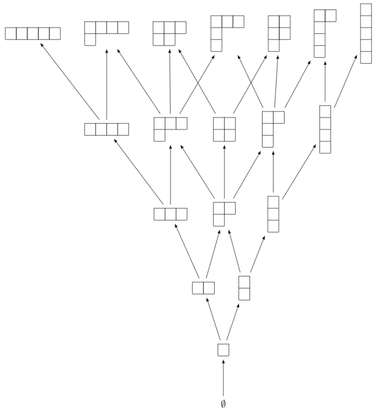


Figure: The Hasse diagram of Young's lattice Y up to rank 5.

Young's lattice Y is the poset of *integer partitions*, non-increasing ordered tuples $\lambda = (\lambda_1, \ldots, \lambda_n)$. These are represented visually by upper-left justified sets of boxes.

An element of Y is greater than another element of Y if each row is at least as large as the equivalent row in the other element.

Differential posets

Definition (Stanley)

An *r-differential poset* P is a poset satisfying the following:

1. P is locally finite, graded, and has a unique minimal element $\widehat{O}$.
2. For every two elements $x, y \in P$, the number of elements covering both x and y is the same as the number of elements covered by both x and y.
3. If an element $x \in P$ covers d elements, then $r + d$ elements cover x.

Example: Young's lattice

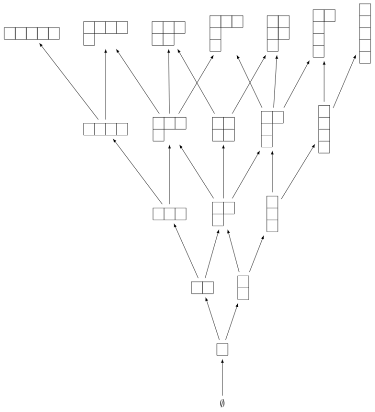


Figure: The Hasse diagram of Young's lattice Y up to rank 5.

Young's lattice Y is a 1-differential poset. Y^r is the r-differential poset form of Young's lattice, which is the set $\underbrace{Y \times Y \times Y \times \ldots \times Y}_{r \text{ times}}$. An element in Y^r is an ordered r-tuple of elements of Y. Stanley conjectured that Y^r is the smallest r-differential poset by size.

Example: Fibonacci Lattices

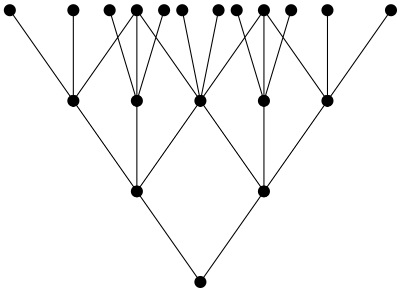


Figure: The Hasse diagram of the Fibonacci lattice $Z(2)$, a 2-differential poset, up to rank 3.

The r-Fibonacci poset, notated by $Z(r)$, is the differential poset defined by the reflection-extension construction.

Fibonnaci Reflection-Extension Construction

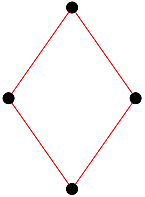


Figure: Reflecting the element in row 0 onto row 2

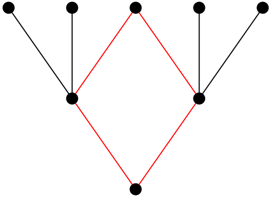


Figure: Extending every element of row 1 twice

Fibonnaci Reflection-Extension Construction

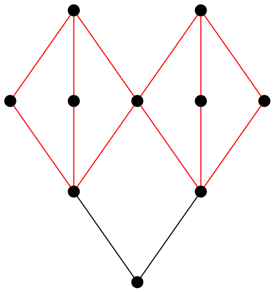


Figure: Reflecting row 1 onto row 3

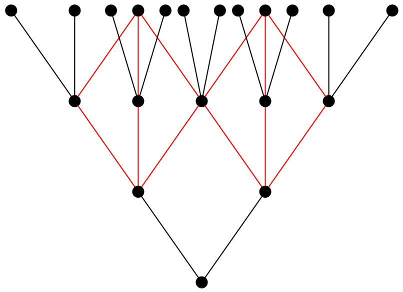


Figure: Extending each element in row 2 twice

Enumerative identities

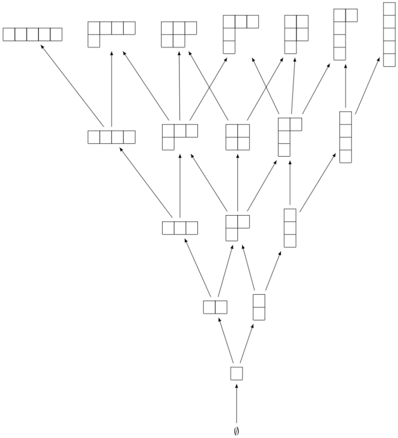
Definition

Define $e(x) = \sum_{y \lessdot x} e(y)$. Equivalently, $e(x)$ equals the number of paths up from $\widehat{O}$ to x.

Many combinatorial and enumerative properties of Young's lattice apply to differential posets in general, making them interesting to study.

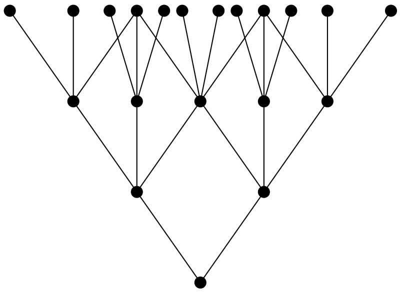
For example, the Robinson-Schensted bijection applied to Young's lattice tells us that $\sum_{x \in P_n} e(x)^2 = n!$ for $x \in Y$. However, $\sum_{x \in P_n} e(x)^2 = r^n n!$ for any r-differential poset P.

Enumerative Identities Example: Young's Lattice



The $e(x)$'s for the elements of row 5 of Y are 1, 4, 5, 6, 5, 4, 1. Therefore, $\sum_{x \in Y_5} e(x)^2 = 1^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 1^2 = 120 = 1^5 * 5!$

Enumerative Identities Example: 2-Fibonacci Poset



The $e(x)$'s for the elements of row 3 of $Z(2)$, the 2-differential Fibonacci poset, are $1, 1, 1, 4, 1, 2, 2, 1, 4, 1, 1, 1$. Therefore, $\sum_{x \in Z(2)_3} e(x)^2 = 1+1+1+16+1+4+4+1+16+1+1+1 = 48 = 2^3 * 3!$

Rank Sizes in Differential Posets

Definition

The *rank* of an element in a differential poset is the number of steps taken to reach $\widehat{O}$.

Definition

Define p_n to be the number of elements in rank n of a differential poset P.

r-Fibonacci Numbers

Definition

The *r-Fibonacci numbers* $F_r(x)$ satisfy $F_r(0) = 1$, $F_r(1) = r$, and $F_r(x) = r \cdot F_r(x-1) + F_r(x-2)$.

Note that if $r = 1$, we just get the regular Fibonacci numbers. Since the reflection-extension construction of the r-Fibonacci poset consists of reflecting the second to last row, and extending r elements per element in the last row, the rank sizes of the r-Fibonacci poset are indeed the r-Fibonacci numbers.

Byrnes' Theorem

Theorem (Byrnes 2012)

For any r-differential poset P we have:

$$p_n \leq r \sum_{i=0}^{n} p_i - (p_{n-1} - 1),$$

and therefore $p_n \leq F_r(n)$.

The r-Fibonacci numbers satisfy Byrnes' inequality, and some induction is sufficient to show $F_r(n)$ is the maximum rank size of rank n.

Uniqueness of the maximal extension

Now, we move on to new results:

Theorem

In a differential poset P, if $p_n = F_r(n)$ for some particular n, then the partial r-differential poset $P_{[0,n]}$ is isomorphic to the r-Fibonacci poset $Z(r)_{[0,n]}$.

Future directions

From the fact that the Fibonacci poset is the largest differential poset, Byrnes hypothesized that the reflection-extension construction will also give the maximal extension for *any* partial differential poset. Equivalently:

Conjecture (Byrnes 2012)

In a differential poset,

$$p_n \leq rp_{n-1} + p_{n-1}$$

Acknowledgements

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Are there any questions?