PRIMES Math Problem Set

PRIMES 2019

Due December 1, 2018

Dear PRIMES applicant:

This is the PRIMES 2019 Math Problem Set. Please send us your solutions as part of your PRIMES application by **December 1, 2018**. For complete rules, see `http://math.mit.edu/research/highschool/primes/apply.php`

Note that this set contains two parts: "General Math problems" and "Advanced Math." Please solve as many problems as you can in both parts.

You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF file. The name of the attached file must start with your last name, for example, "`smith-solutions`". Include your full name in the heading of the file.

Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.

You are allowed to use any resources to solve these problems, *except other people's help*. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

Note that posting these problems on problem-solving websites before the application deadline is strictly forbidden! Applicants who do so will be disqualified, and their parents and recommenders will be notified.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps several days. We encourage you to apply if you can solve at least 50% of the problems.

We note, however, that there will be many factors in the admission decision besides your solutions of these problems.

Enjoy!

General Math Problems

Problem G1. We flip a fair coin ten times, recording a 0 for tails and 1 for heads. In this way we obtain a binary string of length 10.

(a) Find the probability there is exactly one pair of consecutive equal digits.

(b) Find the probability there are exactly n pairs of consecutive equal digits, for every $n = 0, \dots, 9$.

Problem G2. For which positive integers p is there a nonzero real number t such that

$$t + \sqrt{p} \quad \text{and} \quad \frac{1}{t} + \sqrt{p}$$

are both rational?

Problem G3. Points A and B are two opposite vertices of a regular octahedron. An ant starts at point A and, every minute, walks randomly to a neighboring vertex.

(a) Find the expected (i.e. average) amount of time for the ant to reach vertex B.

(b) Compute the same expected value if the octahedron is replaced by a cube (where A and B are still opposite vertices).

Problem G4. For a positive integer n, let $f(n)$ denote the smallest positive integer which neither divides n nor $n + 1$.

(a) Find the smallest n for which $f(n) = 9$.

(b) Is there an n for which $f(n) = 2018$?

(c) Which values can $f(n)$ take as n varies?

Problem G5. A pile with $n \geq 3$ stones is given. Two players Alice and Bob alternate taking stones, with Alice moving first. On a turn, if there are m stones left, a player loses if m is prime; otherwise he/she may pick a divisor $d \mid m$ such that $1 < d < m$ and remove d stones from the pile.

(a) Which player wins for $n = 6$, $n = 8$, $n = 10$?

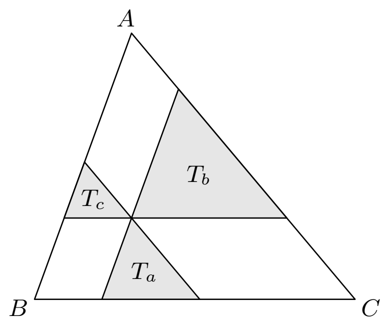
(b) Determine the winning player for all n.

Problem G6. A perfect power is an integer of the form b^n, where $b, n \geq 2$ are integers. Consider matrices 2×2 whose entries are perfect powers; we call such matrices *good*.

(a) Find an example of a good matrix with determinant 2019.

(b) Do there exist any such matrices with determinant 1? If so, comment on how many there could be. (Possible hint: use the theory of Pell equations.)

Problem G7. We consider a fixed triangle ABC with side lengths $a = BC$, $b = CA$, $c = AB$, and a variable point X in the interior. The lines through X parallel to $\overline{AB}$ and $\overline{AC}$, together with line $\overline{BC}$, determine a triangle T_a. The triangles T_b and T_c are defined in a similarly way, as shown in the figure.



Let S and p denote the average area and perimeter, respectively, of the three triangles T_a, T_b, T_c.

(a) Determine all possible values of S as X varies, in terms of a, b, c.

(b) Determine all possible values of p as X varies, in terms of a, b, c.

Advanced Math Problems

Problem M1. Let $\alpha = \sqrt{2} + \sqrt{3}$ and let $V = \mathbb{Q}(\alpha)$ be the field generated by α over $\mathbb{Q}$, regarded as a $\mathbb{Q}$-vector space. Let $T\colon V \to V$ be given by multiplication by α.

(a) Find $\dim V$.

(b) Let $W = \sqrt{2}\mathbb{Q} \oplus \sqrt{3}\mathbb{Q}$. Show that $V = W \oplus T(W)$. Give a basis of $T(W)$.

(c) Compute the determinant of T.

Problem M2. Let n be a positive integer. We denote by I_n the $n \times n$ identity matrix. Let G be a group of $n \times n$ matrices with real entries and determinant 1 (under matrix multiplication).

Suppose that any sequence of matrices in G which converges to I_n is eventually constant. Show that for any $A > 0$, the subset of G with entries in $[-A, A]$ is finite.

Problem M3. (a) If $d \geq 0$ is an integer, evaluate

$$\lim_{n\to\infty} \int_{[0,1]^n} \left[\frac{x_1^2 + \cdots + x_n^2}{n}\right]^d \mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

(b) Evaluate

$$\lim_{n\to\infty} \int_{[0,1]^n} \cos\left[\frac{x_1^2 + \cdots + x_n^2}{n} \cdot \pi\right] \mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

Problem M4. Let n be a fixed positive integer. We choose positive integers $t_1, \ldots, t_n$ (not necessarily distinct) and for each integer r, we let a_r denote the number of subsets $I \subseteq \{1, \ldots, n\}$ for which $\sum_{i\in I} t_i = r$ (this includes $I = \varnothing$ when $r = 0$). Consider the sum

$$\sum_{r\in\mathbb{Z}} a_r^2.$$

(a) Find the minimum possible value of this sum over all choices of $(t_1, \ldots, t_n)$, as a function of n.

(b) Find the maximum possible value of this sum over all choices of $(t_1, \ldots, t_n)$, as a function of n. (Possible hint: Sperner's theorem.)

Problem M5. Exhibit a function $s\colon \mathbb{Z}_{>0} \to \mathbb{Z}$ with the following property: if a and b are positive integers such that $p = a^2 + b^2$ is an odd prime, then

$$s(a) \equiv a^{\frac{p-1}{2}} \pmod{p}.$$

The right-hand side is known as the *Jacobi symbol* $\left(\frac{a}{p}\right)$.

Problem M6. Let G be a nontrivial finite group. We consider automorphisms of G which do not preserve any nontrivial subgroup of G. (An automorphism *preserves* a subgroup of G if the image of that subgroup is itself.)

(a) Determine for which abelian groups G such an automorphism exists.

(b) Find the number of such automorphisms for each such G.

(c) Show that no such automorphisms exist if G is solvable but not abelian.

(d) Generalizing (c), prove that no such automorphisms exist if G is not abelian.