A Penney for Our Thoughts

The Story of a Series of Games and an Avaricious Duo

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Warning! Cheesy puns dead ahead!

Introduction

A Game Coincerning Penneys

This is the original coinflict our current research cleverly cointinues off of:

Alice and Bob possess a stack of penneys they coinnot settle on how to split.

To put a stop to the penney problem for once and for all, they decide to play the one and only...

Penney's Game!!!!

(fanfare plays)

A Game Coincerning Penneys (Cointinued)

They each select a string of heads and tails of a given length n.

For example, if n = 3:

Alice could pick **HHH**:

Bob could pick **THH**:

A Game Coincerning Penneys (Coincluded)

Then, they totally start tossing a terribly troubling penney and cointinue until the sequence of tosses cointains one of their selected sequences:

H H T H **T H H**

YIPPEE!!!

Bob's string **T H H** of penney flips appears before Alice's string **H H H** does, so he wins!!

MUAHAHAHAHA!

A Fairly Fair Argument

For some reason, the friends find from Alice keeping HHH and Bob keeping THH as their strings and then playing Penney's Game plenty of times…

Alice's winnings

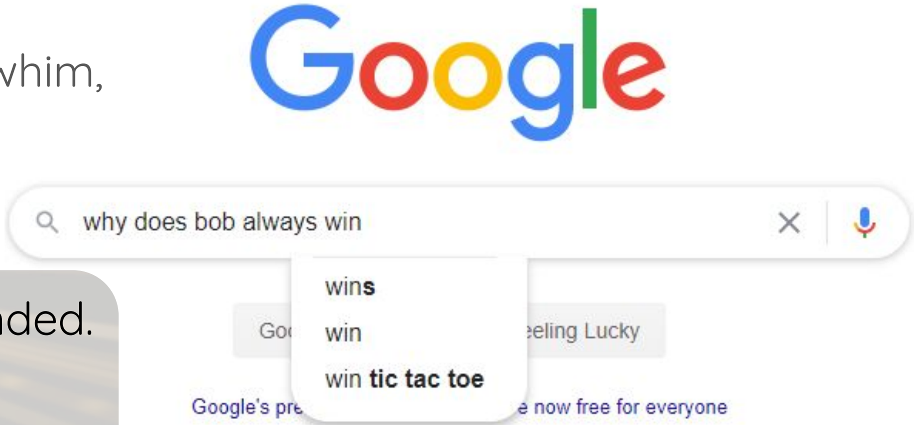
Bob's winnings

…Bob **coin**tinuously won around 7 times as much as Alice did!

A Fairly Fair Argument (Coincluded)

"NAY," Alice exclaimed as Bob lay claim to yet another penney,
"for this simply cannot be!
"Each sequence has an equal $\frac{1}{8}$ chance of occurring, you see?"

"Aye, it is so," Bob replied, "But by fanciful whim,
it seems your chances are slim!
Why is this so?!"



Billowing from the heavens, Google descended.
"Do not welp," it intoned, "for I shall help."
With Google's aid, Alice and Bob reached a
shiny **coin**clusion.

Why the Fairness Argument Doesn't Fair Very Well

After a completely comprehensively **coin**versing curtly with Google, Alice and Bob had figured out that the game was **NOT** fair, along with the reasoning behind it.

But why?

Why the Argument Doesn't Fair Very Well (Cointinued)

The string THH isn't *self-overlapping*: THHTHH

If, perchance, Bob's sequence is rudely "interrupted"...

H T **T H** T

...in truth, his progress is never gutted!

**If only Bob got an H.
If only.**

Why the Argument Doesn't Fair Very Well (Cointinued)

Alice's seemingly sassy string HHH is certainly *self-overlapping*: H**HH**H

If Alice's treacherous penn**ey** resorts to "interrupting"...

H H T

...she must start again from the very beginning!

Why the Argument Doesn't Fair Very Well (Coincluded)

So, say snide Alice's sequence is stopped at some point by an interrupting T:

H H T

Now, she can't win without giving Bob a win first:

H H **T H H**

Alice must attain this or suffer bitter defeat

She needs to get her sequence HHH right away in every game, or Bob will win. Bob clearly has a much higher probability of winning!

Pathways, Driveways, and Coinways

The long-awaited liberating leap:

Conway leading numbers (developed by John H. Conway)

Coinway leading numbers, denoted C_a for a string **a**, are measurements of how self-overlapping a string is!

C_{HHH}: 7 C_{HHT}: 4

C_{HTH}: 5 C_{HTT}: 4

Oh, the many ways and weighs of life...

I Expected to Have to Wait Less, Dear

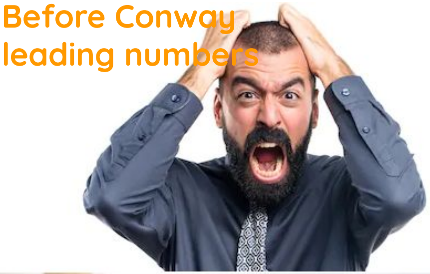
The ***expected wait time*** of a serenely selected string is simply stated as the average number of flips that occur before the string first appears completely in the series of tosses.

Calculating it correctly could covertly cause a cantankerously calamitously catastrophic carnival if not for coinvenient **Coin**way leading numbers:

$$E_a = 2C_a$$

So, $E_{HTT} = 2C_{HTT} = 2(4) = 8$.

Us trying to find expected wait times:



I Expected to Have to Wait Less (Cointinued)

“Aha! Strings bearing greater expected wait times appear last. So, the other string is best!” Bob exclaimed.

Sadly, he is incorrect

For HHH vs HHT...

Expected Wait Times:
HHH: 14
HHT: 8

Both strings need HH, but the flip after will determine the winner. If it is H, then HHH wins, and if it is T, then HHT wins. This is a ½ chance, even though HHH has a higher expected wait time.

Bob's Optimal Solution

The game is unfair. Bob can always choose a sequence that gives him an advantage over Alice.

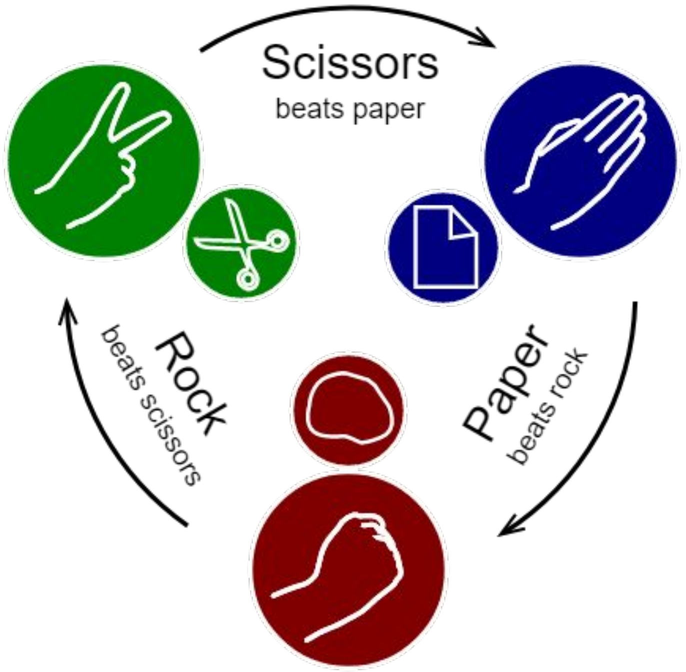
Alice's string	Bob's best choice	Bob's odds
HHH	THH	7 to 1
HHT	THH	3 to 1
HTH	HHT	2 to 1
HTT	HHT	2 to 1

A table of Bob's best choice for strings of length 3

The philosophy of Bob's life

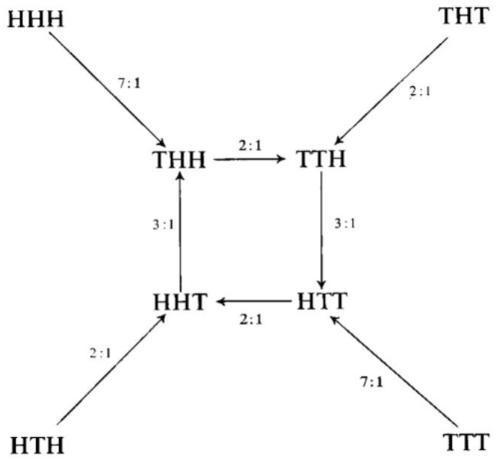
Transitive vs Non-transitive

$$a > b > c \Rightarrow a > c$$



The Non-transitive Cycle

Alice and Bob carefully calculated all the odds for all of the games and simply stumbled upon a certainly seriously stunning *non-transitive cycle*:



THH loses to TTH, TTH loses to HTT, HTT loses to HHT and HHT loses to THH. It's like a game of rock-paper-scissors.

Our Cointinuation

Alice Kinda Loses It

“This is simply not fair for either in this pair,” Alice cried out of frustration.

Bob proposed a potential solution: “What if you began with an extra toss? Will it alleviate this equality loss?”

And so the two tried this new game— but will it shift the odds enough to make them even?

Head-Start Penney's Game

Alice and Bob attempt this new game, where Alice gets an extra toss at the beginning.

- First extra toss only gives Alice an advantage when she gets her string of length n in the first n tosses
- Otherwise, this game becomes Penney's game
- Bob's optimal solution is the same
- String with the longer expected wait time can still win
- Bob still has an advantage over Alice, but the odds are better

Post-aBOBalyptic Game

Alice, furious at how the game wasn't fair, devised a new idea. What if she got an extra toss, but it only counted at the end? In other words, if Bob won, Alice would get one final toss to try and tie things up.

As long as the last n-1 elements of Bob's string aren't the same as Alice's first n-1 elements, the probabilities are the same as the normal Penney's game.

Otherwise, they tie with probability ½

Alice	Bob	Odds
HHH	THH	7 to 2
HHT	THH	3 to 2
HTH	TTH	5 to 3
HTT	HHT, HTH, THH	1 to 1

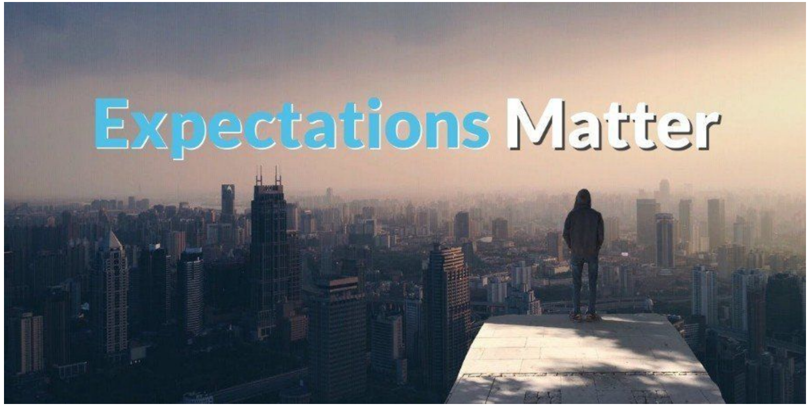
The New Odds

A Game With Two Coins

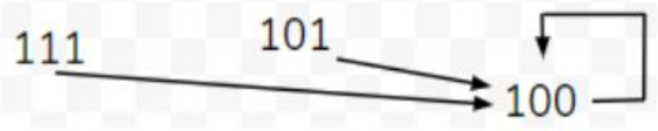
- The situation is as follows:
  - They now each have their own coin
  - Each turn they toss their coins simultaneously
  - Whoever gets their chosen sequence of tosses first wins

A Game With Two Coins: An Overview

- The probability of winning at each turn can be summed to get the final probability of winning
- It suffices to say that this game is completely computationally complicated
- Notable observations:
  - A tie is now possible
  - The win/tie/loss probabilities are based only on expected wait time



Wait, expected wait times matter now...no one expected that coming



A Game With Two Coins: The Coinvincing Reason

- There isn't really much to say, we used a computer to find these:

	HHH	HTH	THH
HHH	$\frac{435}{913}, \frac{435}{913}, \frac{43}{913}$	$\frac{3289}{8691}, \frac{1643}{2897}, \frac{473}{8691}$	$\frac{23327}{73057}, \frac{45409}{73057}, \frac{4321}{73057}$
HTH	$\frac{1643}{2897}, \frac{3289}{8691}, \frac{473}{8691}$	$\frac{487}{1045}, \frac{487}{1045}, \frac{71}{1045}$	$\frac{22431}{55265}, \frac{28673}{55265}, \frac{4161}{55265}$
THH	$\frac{45409}{73057}, \frac{23327}{73057}, \frac{4321}{73057}$	$\frac{28673}{55265}, \frac{22431}{55265}, \frac{4161}{55265}$	$\frac{377}{825}, \frac{377}{825}, \frac{71}{825}$

Bob wins at most ½ of the time for all of these

- (In order, these are the probabilities of Alice winning, Bob winning, and a tie)

Second-Occurence Game

- Logically, there will be more even odds
  - Self-repeating strings do better
  - For example HHH vs THH will have odds of 11/16 instead of 14/16

The odds are more equitable

A Strange Case

- Coinsider the case HHT vs THH
  - After you get HHT, you must get THH next
  - After you get THH, you must get HHT next

HHT HHT

HHTHHT

Coinclusion

- Now, Alice only loses with odds of about 60%, instead of 66%
    - Alice’s best choices are HTH and HTT

No-Flippancy Game

No-Flippancy Game
Introduction

Alice and Bob look at their sequences, pick the letter that they need to progress forward. Alternate turns.

Alice - HTH and Bob - THH

Alice string = Bob string

Possible Outcomes:

Alice wins - Bob wins - Alice/Bob tie - Infinite

No-Flippancy Game
Definitions

Forced Outcome:
One player picks their string; other wants particular outcome.

Alternating String:
HTHTHTHT

Complementary String:
HTH; THT
HHHTHTH; TTTHTHT

No-Flippancy Game
Summary

Forcing Outcomes:

Bob can force infinite games unless Alice has H, HT, HTH, HTHT, or their complements.

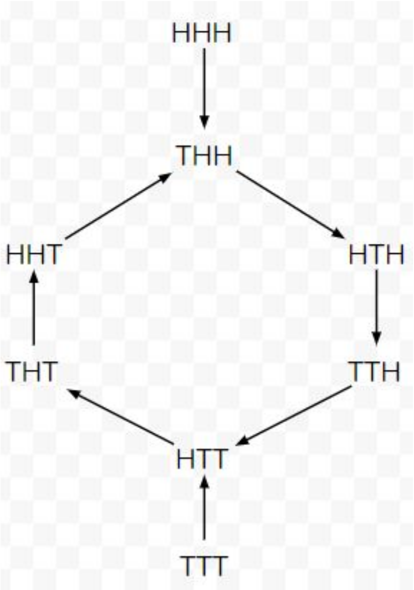
Alice can force infinite games unless Bob has H, HT, HTH, HTHT, HTHTH, or their complements.

Bob can force a win for himself unless Alice has an H or T.

Alice can always force a win.

Blended Game

- If Alice and Bob want the same letter, then they will get it
- Logically, Bob should pick a string with the opposite first letter of Alice, followed by Alice's first two letters
  - HHH -> THH
- If Alice picks HTH, at best Bob can win with at most ⅝ probability



A picture of Bob's best choice given Alice's string

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AND FINALLY,
VIEWERS LIKE YOU
THANK YOU!

QUESTIONS?

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