Game Theory: A Playful Presentation

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Overview

Combinatorial Game Theory

Definition

A *combinatorial game* is a 2-player game played between Louise and Richard. The game consists of the following:

1. A set of possible *positions*, or the states of the game.
2. A *move rule* indicating for each position what positions Louise can move to and what positions Richard can move to.
3. A *win rule* indicating a set of *terminal positions* where the game ends. Each terminal position has an associated *outcome*, either Louise wins and Richard loses, Louise loses and Richard wins, or it is a draw.

Example

In the game Tic, there is a 1×3 array. To move, Louise marks an empty square with a $\circ$ and Richard with a $\times$. If either player gets two adjacent squares marked with his or her symbol, then they win.

Categorizations of Games

Definition

- A *normal-play* game is a combinatorial game such that the last person to make a move wins.
  - Example: Checkers.
- An *impartial game* is a normal-play game where the available moves from every position are always the same for either player.
  - Example: Tic.
- A *partizan game* is a normal-play game where the available moves for Louise and Richard are not the same.
  - Example: Chess.

Classifications of Positions

- **Type L**: Louise has a winning strategy no matter who goes first.
- **Type R**: Richard has a winning strategy no matter who goes first.
- **Type N**: The Next player has a winning strategy.
- **Type P**: The Previous or second player has a winning strategy.

Rules for Beads

Definition

Beads is a normal-play, impartial game that has 3 different movement options during a player's turn.

1. The player can remove 1 bead
2. The player can remove 2 beads
3. When the amount of beads on the chain is even, the player can choose to split the chain into 2 equivalent chains

When there are multiple chains, the player can only remove beads from one chain each turn.

Example

There is a chain with 4 beads. Louise will go first, and she can choose to remove 1 bead, 2 beads, or split the chain into 2 equivalent chains that are 2 beads long. The last player to take a bead wins.

Examples

Example

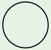


Figure: A 1 bead chain is Type N.

Example

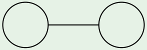


Figure: A 2 bead chain is also Type N

Example

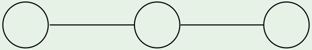


Figure: A chain with 3 beads is Type P. The next player will leave either 1 or 2 beads which are both Type N positions

Mirror Lemma

Lemma

A position of the form $\alpha + \alpha$ in an impartial game is Type P.

Proof.

Case 1: α is Type N or P. Any move that Player 1 makes can be mirrored by Player 2 on the opposite chain, ultimately leading to Player 2 winning the game. $\square$

Example

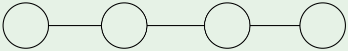
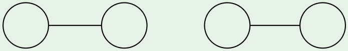


Figure: There are 3 options for the player from this Type N position: 1. Split in half, 2. Remove 1 bead, or 3. Remove 2 beads.



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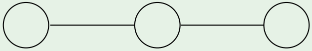

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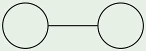


Figure: The 2 bead chains are Type P by the Mirror Lemma, The 3 bead position is Type P, and the 2 bead position is Type N.

Example

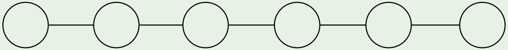


Figure: A chain with 6 beads

Louise will go first in this game. She can remove 1 bead, 2 beads, or split the chain in half. Louise will split the chain.

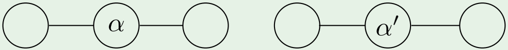


Figure: The 2 identical chains, α and α'

From here, Louise can copy whatever move Richard does on the opposite chain, guaranteeing her the last move by the Mirror lemma.

The Pattern in the Games

Proposition

# of Beads	1	2	3	4	5	6	7	8	9	10	11
Position Type	N	N	P	N	N	N	P	N	N	N	P

General Proof/Theorem

Theorem

If $n \equiv 3 \pmod 4$, *then the n bead game is Type P. Otherwise, when* $n \not\equiv 3 \pmod 4$, *the game is Type N.*

- $n \equiv 3 \pmod 4$ is a number that will equal 3 when 4 is continuously subtracted from the original number. This number will always be odd.
- $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$ are numbers that will always be even, as well as the concept above. The winning strategy for these amounts is for the next player to split the chain in half, making it Type N.
- $n \equiv 1 \pmod 4$ is an odd number where the winning strategy is for the next player to remove 2 beads, creating a $n \equiv 3 \pmod 4$ position that is Type P.
- For example, 5 can be our number n. This is an where $n \equiv 1 \pmod 4$. By removing 2 beads, we get 3 beads, which is $n \equiv 3 \pmod 4$; this is a Type P position.

Rules for Traveling

- Traveling is played between Louise and Richard on a $k \times \ell$ grid.
- On Louise's turn, she can move the token to the left or down as many edges as she wants, staying on the grid. On Richard's turn, he can move to the right or up as many edges he wants, staying on the grid.
- The first player to move the token to the bottom right corner wins.

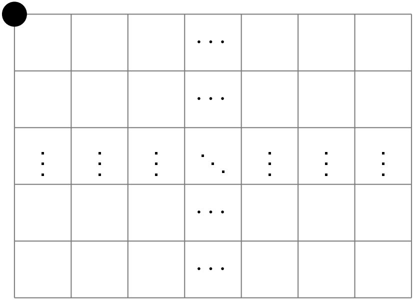


Figure: A starting position in a $k \times \ell$ game of Traveling

General Strategy

- For any game of Traveling, Louise wants to force Richard to reach the rightmost edge so she can go straight down, while Richard wants to force Louise to the bottommost edge.
- For this reason, Louise does not want to move downwards and Richard does not want to move rightwards.

A Basic Game of Traveling

Example

Let Louise's moves be red and Richard's moves be blue.

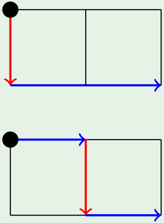
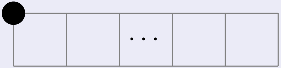


Figure: A 1×2 game of Traveling

Simple Cases of Traveling

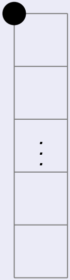
Theorem

Any $1 \times k$ game of Traveling, where $k > 1$, is type R.



Theorem

Any $k \times 1$ game of Traveling, where $k > 1$, is type L.



Symmetry in Traveling

The similarity between the analysis of the $1 \times k$ and $k \times 1$ games of Traveling reflects a more general symmetry between $k \times \ell$ and $\ell \times k$ games.

Theorem

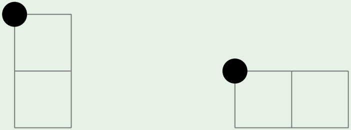
For positive integers k and ℓ, the following hold:

1. *If a $k \times \ell$ game of Traveling is type P, then an $\ell \times k$ game is also type P.*
2. *If a $k \times \ell$ game of Traveling is type N, then an $\ell \times k$ game is also type N.*
3. *If a $k \times \ell$ game of Traveling is type L, then an $\ell \times k$ game is type R.*
4. *If a $k \times \ell$ game of Traveling is type R, then an $\ell \times k$ game is type L.*

Examples of Symmetry

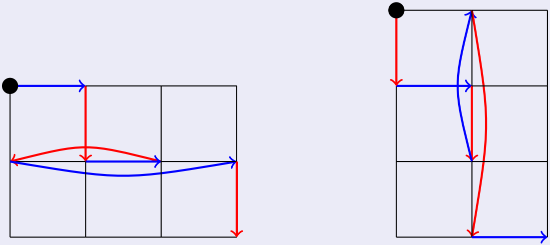
Example

Consider a 2×1 game of Traveling vs. a 1×2 game.



Proposition

A 3×2 game of Traveling (and thus, a 2×3 game) is type P.

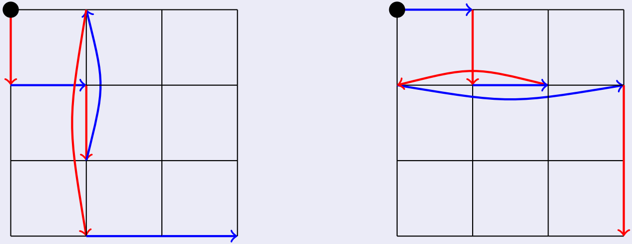


Symmetry in $n \times n$ Games

While we have established that a $k \times \ell$ game is symmetric to an $\ell \times k$ game, a $k \times k$ game is symmetric to itself.

Proposition

A 3×3 game of Traveling is type P.



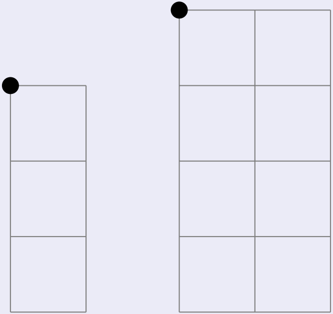
Corollary

An $k \times k$ game of Traveling is either type N or type P.

Additional Ideas

Conjecture

For any $k \times \ell$ game of Traveling, if $k \geq \ell + 2$, the game is type L. if $\ell \geq k + 2$, the game is type R. This excludes the cases in the 2×1 and 1×2 games.



- While we have not solved Traveling completely, the results included here and other examples we have explored gives us some ideas about general good practices.
- When Louise plays, she wants to go down as slowly as she can, choosing to go left whenever possible. The same goes for Richard, who wants to move to the right as slowly as he can.
- The resulting race between Louise and Richard makes a game of Traveling.

References and Acknowledgements

References:

[1] M. DeVos and D. A. Kent. *Game Theory: A Playful Introduction*. *American Mathematical Society* Vol. 80, 2016.

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