Optimal solutions and ranks in the max-cut SDP

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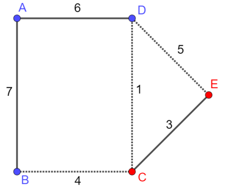
Max-cut Problem

Cut

Consider a graph $G = (V, E)$ with vertex set $V = [n]$ and fixed weights $\{w_{ij}\}_{ij \in E}$ assigned to the edges. A *cut* is a partition of the vertex set $V = V_1 \sqcup V_2$.

Example

This partitions the graph into {C, E} and {A, B, D}, cutting across edges with a total sum of 10.



Max-cut Problem

Max-cut Problem

The *max-cut problem* asks to split the vertex set of a graph into two groups to maximize the sum of edge weights between the two groups. In other words, we wish to find the maximum possible value of a cut $V_1 \sqcup V_2$.

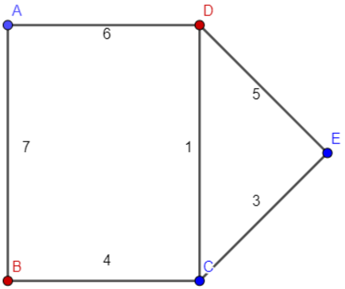
Applications

Applications of the max-cut problem:

- Theoretical/Statistical physics
- Circuit design

Max-cut Problem

Example



Above shows the max cut possible for this graph; it has a value of 23.

Max-cut Problem

Properties of max-cut problem

- The max-cut problem is *NP-complete*.
- However, the problem can be approximated in polynomial time up to a factor of 0.87854.
- This uses a technique known as *semidefinite programming* (SDP).

Representing Graphs

Laplacian Matrix

Consider a graph $G = (V, E)$ with vertex set $V = [n] = \{1, 2, \ldots, n\}$ and weights w. We define the *Laplacian matrix* $L(G, w)$ as the $n \times n$ matrix with entries

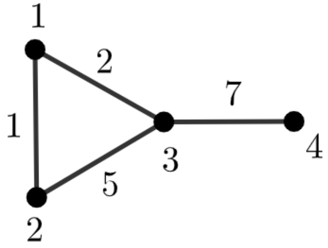
$$L(G,w)_{ij} := \begin{cases} -w_{ij} & \text{if } ij \in E \\ \sum_k w_{ik} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Representing Graphs

Example

The Laplacian matrix of the graph below is

$$\begin{bmatrix} \mathbf{3} & -1 & -2 & 0 \\ -1 & \mathbf{6} & -5 & 0 \\ -2 & -5 & \mathbf{14} & -7 \\ 0 & 0 & -7 & \mathbf{7} \end{bmatrix}.$$



PSD Matrices and Semidefinite programming

Positive Semidefinite Matrices

A d-by-d symmetric matrix M is *positive semidefinite* (PSD), or $M \succeq 0$, if and only if there exists a square root matrix B such that $B^T B = M$.

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Frobenius Inner Product

$\bullet$ represents the Frobenius inner product, which is the entry-wise product summed over all entries.

$$\begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \bullet \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} = (2)(-1) + (2)(0) + (0)(2) + (3)(1) = 1.$$

Semidefinite Programs

Semidefinite Program

Let C be a $n \times n$ cost matrix. Consider m constraint matrices $A_1, A_2, \ldots, A_m \in \mathbb{S}^n$, as well as a constraint vector $b \in \mathbb{R}^m$. A *semidefinite program* is an optimization problem of the form

$$\begin{aligned} \max_{X \in \mathbb{S}^n} \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = b_i \quad \forall 1 \leq i \leq m \\ & X \succeq 0 \end{aligned}$$

Note that this optimization is *linear* in the entries of X. In fact, there are known algorithms to solve it in polynomial time.

Example of an SDP

Example

$$\begin{aligned} \max_{X \in \mathbb{S}^n} \quad & X \bullet \begin{bmatrix} 1 & 8 \\ 8 & -1 \end{bmatrix} \\ \text{s.t.} \quad & X \bullet \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 \\ & X \succeq 0. \end{aligned}$$

Given $X = \begin{bmatrix} z & y \\ y & x \end{bmatrix}$, the constraints become $z = 1$ and $x \geq y^2$, and we must maximize $1 + 16y - x$. The maximum is $\boxed{65}$.

Primal Max-cut SDP

Primal max-cut SDP

The following is the SDP relaxation of the max-cut problem:

$$\begin{aligned} \max_{X \in \mathbb{S}^n} \quad & \frac{1}{4} L(G, w) \bullet X \\ \text{s.t.} \quad & X_{ii} = 1 \text{ for } i \in [n] \\ & X \succeq 0. \end{aligned}$$

We denote the optimal primal solution by $\bar{X}$.

Dual Max-cut SDP

Dual max-cut SDP

The following is the dual of the primal max-cut SDP:

$$\begin{aligned} \min_{y \in \mathbb{R}^n, S \in \mathbb{S}^n} \quad & \sum y_i \\ \text{s.t.} \quad & S = \mathrm{Diag}(y) - C \\ & S \succeq 0. \end{aligned}$$

We similarly denote the dual optimal solution by $\bar{S}$.

Rank of Max-cut SDP

Primal max-cut SDP

$$\begin{aligned} \max_{X \in \mathbb{S}^n} \quad & \frac{1}{4} L(G, w) \bullet X \\ \text{s.t.} \quad & X_{ii} = 1 \text{ for } i \in [n] \\ & X \succeq 0. \end{aligned}$$

- When we write the max-cut problem algebraically, a vector $x \in \mathbb{R}^n$ represents a cut, and we write $X = xx^T$.
- The condition $X = xx^T \iff \operatorname{rank} X = 1$. Furthermore, all symmetric rank-1 matrices are positive semidefinite.
- All rank-1 primal optimal solutions to the SDP must be exact solutions to the max-cut problem.

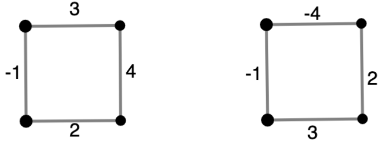
Rank-1 solutions for cycles

Theorem (Rank 1 solutions of cycles)

The cycle graph exhibits a rank 1 solution if and only if at least one of the following holds:

- *There are an even number of positively weighted edges*
- *Take the list of the absolute values of the reciprocal of every edge weight (with repetition). Then there is a value in this list that is at least the sum of the rest.*

- Rank 1 solutions: $\{-4, 2, 3, -1\}, \{-2, -3, 8\}$.
- Rank 2 solutions: $\{4, 2, 3, -1\}, \{-2, -3, 5\}$.



Max-Cut Problem on cycles

- Ideally, the cut of a graph should split the edges with positive weights and not split those with negative weights.
- Using this, we can attempt to do this for every edge. From this, we can solve for the best solution for the max-cut SDP.

Motivating theorem

This theorem is presented without proof:

Theorem (Primal-dual feasibility for optimal solutions)

Let matrices $\bar{X}$ and $\bar{S}$ be optimal primal and dual matrices for the max-cut problem on a graph G, respectively. Then $\bar{X}\bar{S} = 0$ (all entries are 0).

This theorem tells us that every column of $\bar{S}$ is in the nullspace of every column of $\bar{X}$. Since $\bar{S}$ is known on all off-diagonal values, given a $\bar{X}$, one can find all entries of $\bar{S}$ (and $\bar{S}$ is thus unique).

Identifying rank 1 solutions using $\bar{S}$

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Identifying rank 1 solutions using $\bar{S}$

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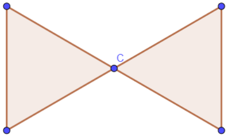
- Identify possible rank 1 primal optimal matrix solutions to the max-cut problem
- Identify possible dual matrices resulting from these primal matrices
- Impose the positive semi-definite constraint.

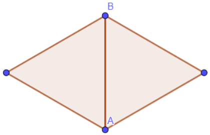
Ranks and Solutions for Clique Sums

We provide two examples of clique sums for illustration.

The butterfly graph below is the *vertex sum* of two K_3 graphs joined at vertex C.



The diamond graph below is the *edge sum* of two K_3 graphs joined at edge AB.



Ranks and Solutions for Clique Sums

Given two graphs and optimal solutions to their max-cut SDP, we show that we can *find an optimal solution* to the max-cut SDP of their *vertex sum*.

Given two K_3 graphs, we can find an optimal solution *given certain conditions on the weights* in terms of the optimal solutions of the K_3 graphs' SDPs.

Future work:

- General result for edge sums
- Extension to more families of graphs

Acknowledgements

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- Our parents