Few Distance Sets in ℓ_p Spaces and ℓ_p Product Spaces

Richard Chen, Jason Tang, Nathan Xiong
Mentor: Feng Gui

October 17-18, 2020
MIT PRIMES Conference

Introduction

Definition

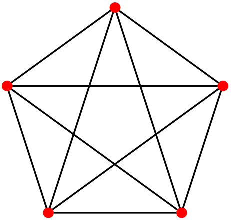
An **s-distance set** is a set of points such that the set of pairwise distances between the points has cardinality s.

Introduction

Definition

An **s-distance set** is a set of points such that the set of pairwise distances between the points has cardinality s.

- A regular pentagon is a 2-distance set in $\mathbb{R}^2$:

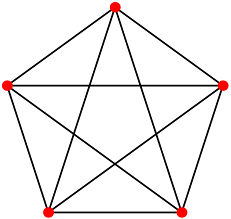


Introduction

Definition

An **s-distance set** is a set of points such that the set of pairwise distances between the points has cardinality s.

- A regular pentagon is a 2-distance set in $\mathbb{R}^2$:



Definition

A 1-distance set is also known as an **equilateral set**.

General Problem

Question

Given a metric space X and a positive integer s, what is the maximum size of an s-distance set in X?

General Problem

Question

Given a metric space X and a positive integer s, what is the maximum size of an s-distance set in X?

Definition

Let $e_s(X)$ denote the answer to this question, or just $e(X)$ if $s = 1$.

Some problems we can consider:

General Problem

Question

Given a metric space X and a positive integer s, what is the maximum size of an s-distance set in X?

Definition

Let $e_s(X)$ denote the answer to this question, or just $e(X)$ if $s = 1$.

Some problems we can consider:

- Different spaces: Euclidean space, sphere, hyperbolic space, etc.

General Problem

Question

Given a metric space X and a positive integer s, what is the maximum size of an s-distance set in X?

Definition

Let $e_s(X)$ denote the answer to this question, or just $e(X)$ if $s = 1$.

Some problems we can consider:

- Different spaces: Euclidean space, sphere, hyperbolic space, etc.
- Different metrics: Euclidean distance, taxicab distance, Hamming distance, etc.

General Problem

Question

Given a metric space X and a positive integer s, what is the maximum size of an s-distance set in X?

Definition

Let $e_s(X)$ denote the answer to this question, or just $e(X)$ if $s = 1$.

Some problems we can consider:

- Different spaces: Euclidean space, sphere, hyperbolic space, etc.
- Different metrics: Euclidean distance, taxicab distance, Hamming distance, etc.
- Different s values: equilateral sets, 2-distance sets, general s-distance sets, etc.

General Problem

Question

Given a metric space X and a positive integer s, what is the maximum size of an s-distance set in X?

Definition

Let $e_s(X)$ denote the answer to this question, or just $e(X)$ if $s = 1$.

Some problems we can consider:

- Different spaces: Euclidean space, sphere, hyperbolic space, etc.
- Different metrics: Euclidean distance, taxicab distance, Hamming distance, etc.
- Different s values: equilateral sets, 2-distance sets, general s-distance sets, etc.
- Almost-equilateral sets: all distances are within ε of d, instead of being exactly d.

Example in Euclidean Space

Example

What is the maximum size of an equilateral set in $\mathbb{E}^n$?

Example in Euclidean Space

Example

What is the maximum size of an equilateral set in $\mathbb{E}^n$?

- The answer is $n+1$, so $e(\mathbb{E}^n) = n+1$. Equality holds with a standard simplex.

Example in Euclidean Space

Example

What is the maximum size of an equilateral set in $\mathbb{E}^n$?

- The answer is $n+1$, so $e(\mathbb{E}^n) = n+1$. Equality holds with a standard simplex.

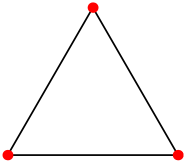


Figure: An equilateral triangle in $\mathbb{E}^2$

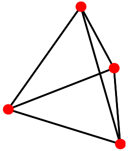


Figure: A regular tetrahedron in $\mathbb{E}^3$

Linear Algebra Proof

Linear Algebra Proof

- Let the equilateral set consist of points $p_1, \ldots, p_m$. Assume that the common distance is 1.

Linear Algebra Proof

- Let the equilateral set consist of points $p_1, \ldots, p_m$. Assume that the common distance is 1.
- For each $1 \leq i \leq m$, define the polynomial

$$f_i(x) = 1 - \|x - p_i\|^2 ,$$

so that $f_i(p_j) = \delta_{ij}$. This implies the f_i are linearly independent.

Linear Algebra Proof

- Let the equilateral set consist of points $p_1, \ldots, p_m$. Assume that the common distance is 1.
- For each $1 \leq i \leq m$, define the polynomial

$$f_i(x) = 1 - \|x - p_i\|^2 ,$$

so that $f_i(p_j) = \delta_{ij}$. This implies the f_i are linearly independent.
- Expand f_i as

$$f_i(x) = \|x\|^2 - \sum_{j=1}^{n} c_j x_j + C.$$

Thus, the f_i live in a vector space of dimension $n+2$ and $m \leq n+2$.

Linear Algebra Proof

- Let the equilateral set consist of points $p_1, \ldots, p_m$. Assume that the common distance is 1.
- For each $1 \leq i \leq m$, define the polynomial

$$f_i(x) = 1 - \|x - p_i\|^2,$$

so that $f_i(p_j) = \delta_{ij}$. This implies the f_i are linearly independent.
- Expand f_i as

$$f_i(x) = \|x\|^2 - \sum_{j=1}^{n} c_j x_j + C.$$

Thus, the f_i live in a vector space of dimension $n + 2$ and $m \leq n + 2$.
- [Blokhuis trick] The stronger claim that $f_1, \ldots, f_m, 1$ are linearly independent ends up giving the desired $m \leq n + 1$.

Strategy

Our strategy for the example before was:

- Define an annihilating function, which happens to be a polynomial.
- Obtain bound from linear independence.

We can use this strategy whenever the annihilating function is a polynomial, e.g., with an s-distance set in $\mathbb{E}^n$.

The ℓ_p norm

Definition

For any $p \geq 1$, the ℓ_p norm is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}.$$

Taking $p \to \infty$, we can define the ℓ_∞ norm by

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

Definition

The space $\mathbb{R}^n$ equipped with the ℓ_p norm is denoted ℓ_p^n.

The ℓ_p sum

Definition

We define the ℓ_p sum $\mathbb{E}^{a_1} \oplus_p \cdots \oplus_p \mathbb{E}^{a_n}$ to be the product space $\mathbb{E}^{a_1} \times \cdots \times \mathbb{E}^{a_n}$ equipped with the norm

$$\|(x_1, \ldots, x_n)\|_p = \left(\sum_{i=1}^{n} \|x_i\|_2^p \right)^{\frac{1}{p}} .$$

If we take $p \to \infty$, the norm becomes

$$\|(x_1, \ldots, x_n)\|_\infty = \max_{1 \leq i \leq n} \|x_i\| .$$

Our First Two Results

Theorem (Chen et. al., 2020)

Let $\mathbb{E}^a$ and $\mathbb{E}^b$ be Euclidean spaces. Then,

$$e(\mathbb{E}^a \oplus_\infty \mathbb{E}^b) \leq (a+1)(b+1)+1.$$

Our First Two Results

Theorem (Chen et. al., 2020)

Let $\mathbb{E}^a$ and $\mathbb{E}^b$ be Euclidean spaces. Then,

$$e(\mathbb{E}^a \oplus_\infty \mathbb{E}^b) \leq (a+1)(b+1)+1.$$

- Annihilating functions: $f_u(x) = \left(1 - \|\widetilde{x}_1 - \widetilde{u}_1\|^2\right)\left(1 - \|\widetilde{x}_2 - \widetilde{u}_2\|^2\right)$.

Our First Two Results

Theorem (Chen et. al., 2020)

Let $\mathbb{E}^a$ and $\mathbb{E}^b$ be Euclidean spaces. Then,

$$e(\mathbb{E}^a \oplus_\infty \mathbb{E}^b) \leq (a+1)(b+1)+1.$$

- Annihilating functions: $f_u(x) = \left(1 - \|\widetilde{x}_1 - \widetilde{u}_1\|^2\right)\left(1 - \|\widetilde{x}_2 - \widetilde{u}_2\|^2\right)$.
- Start with $(a+2)(b+2)$ element spanning set of the f_i.

Our First Two Results

Theorem (Chen et. al., 2020)

Let $\mathbb{E}^a$ and $\mathbb{E}^b$ be Euclidean spaces. Then,

$$e(\mathbb{E}^a \oplus_\infty \mathbb{E}^b) \leq (a+1)(b+1)+1.$$

- Annihilating functions: $f_u(x) = \left(1 - \|\widetilde{x}_1 - \widetilde{u}_1\|^2\right)\left(1 - \|\widetilde{x}_2 - \widetilde{u}_2\|^2\right)$.
- Start with $(a+2)(b+2)$ element spanning set of the f_i.
- We eliminate $a+b+2$ elements from the spanning set via a Blokhuis trick technique.

Our First Two Results

Theorem (Chen et. al., 2020)

Let $\mathbb{E}^a$ and $\mathbb{E}^b$ be Euclidean spaces, and let p be an even integer. Then,

$$e(\mathbb{E}^a \oplus_p \mathbb{E}^b) \leq \binom{a+p/2}{a} + \binom{b+p/2}{b}.$$

- Annihilating functions:
  $f_u(x) = 1 - \left(\sum_{t=1}^{a}(\widetilde{x}_{1t} - \widetilde{u}_{1t})^2\right)^{p/2} - \left(\sum_{t=1}^{b}(\widetilde{x}_{2t} - \widetilde{u}_{2t})^2\right)^{p/2}.$
- We use the same strategy as the previous result.

Conjectures and Known Results

Conjecture (Kusner, 1983)

For $p \in (1, \infty)$, *we have* $e(\ell_p^n) = n + 1$.

Currently, the best bound is

Theorem (Alon-Pudlák, 2003)

For every $p \geq 1$, *we have* $e(\ell_p^n) \leq c_p n^{(2p+2)/(2p-1)}$, *where one may take* $c_p = cp$ *for an absolute* $c > 0$.

Rank Lemma

Definition

The **rank** of a matrix is the dimension of the space spanned by its column (or row) vectors.

Lemma (Rank Lemma)

For a real symmetric $n \times n$ nonzero matrix A,

$$\operatorname{rank} A \geq \frac{\left(\sum_{i=1}^{n} a_{ii}\right)^2}{\sum_{i,j=1}^{n} a_{ij}^2}.$$

Rank Lemma in Action

If the polynomials f_i are not linearly independent, we can take the matrix A with $a_{ij} = f_i(p_j)$ and hope that it looks something like

$$\begin{pmatrix} 1 & & & & \\ & 1 & & \mathcal{E} & \\ & & 1 & & \\ & \mathcal{E} & & \ddots & \\ & & & & 1 \end{pmatrix}$$

By the Rank Lemma, this matrix has large rank in terms of n. We can then upper bound the rank using generators of the polynomials, like in the Euclidean Distance example.

Jackson's Theorem

Jackson's Theorem allows us to approximate a function by a polynomial with sufficiently small error.

Lemma (Jackson's Theorem on $|x|^p$)

For any $p \geq 1$ and $d \geq \lceil p \rceil$, there exists a polynomial P with degree at most d such that

$$|P(x) - |x|^p| \leq \frac{B(p)}{d^p} \qquad \text{for all } x \in [-1, 1],$$

where $B(p) < (cp)^p$ for an absolute $c > 0$.

Strategy

Recall our strategy from the example in Euclidean space:

- Define an annihilating function, which happens to be a polynomial.
- Obtain bound from linear independence.

Strategy

Recall our strategy from the example in Euclidean space:

- Define an annihilating function, which happens to be a polynomial.
- Obtain bound from linear independence.

Theorem (Alon-Pudlák, 2003)

For every $p \geq 1$, we have $e(\ell_p^n) \leq c_p n^{(2p+2)/(2p-1)}$, where one may take $c_p = cp$ for an absolute $c > 0$.

In Alon and Pudlák's case, the annihilating function isn't a polynomial, so the strategy is now:

- Define an annihilating function.
- Approximate this function with a polynomial.
- Obtain bound from rank argument.

Our Results

When p is large in terms of n, we have the stronger result:

Theorem (Chen et. al., 2020)

If $n > 1$ and $p \geq c(n \log n)^2$ for an absolute $c > 0$, then $e(\ell_p^n) \leq 2(p+1)n$.

Our Results

When p is large in terms of n, we have the stronger result:

Theorem (Chen et. al., 2020)

If $n > 1$ and $p \geq c(n \log n)^2$ for an absolute $c > 0$, then $e(\ell_p^n) \leq 2(p+1)n$.

- Consider the space ℓ_k^n, where k is the closest even number to p.

Our Results

When p is large in terms of n, we have the stronger result:

Theorem (Chen et. al., 2020)

If $n > 1$ and $p \geq c(n \log n)^2$ for an absolute $c > 0$, then $e(\ell_p^n) \leq 2(p+1)n$.

- Consider the space ℓ_k^n, where k is the closest even number to p.
- If p is large enough, we can approximate the ℓ_p norm with the ℓ_k norm.

Our Results

When p is large in terms of n, we have the stronger result:

Theorem (Chen et. al., 2020)

If $n > 1$ and $p \geq c(n \log n)^2$ for an absolute $c > 0$, then $e(\ell_p^n) \leq 2(p+1)n$.

- Consider the space ℓ_k^n, where k is the closest even number to p.
- If p is large enough, we can approximate the ℓ_p norm with the ℓ_k norm.
- We instead bound the size of an almost-equilateral set in ℓ_k^n, which can be done with the Rank Lemma.

Our Results

We can also generalise Alon and Pudlák's theorem to s-distance sets.

Theorem (Chen et. al., 2020)

If s is a positive integer and p is a real number satisfying $2p > s$, then $e_s(\ell_p^n) \leq c_{p,s} n^{(2ps+2s)/(2p-s)}$ for a constant $c_{p,s}$ depending on p and s.

Our Results

We can also generalise Alon and Pudlák's theorem to s-distance sets.

Theorem (Chen et. al., 2020)

If s is a positive integer and p is a real number satisfying $2p > s$, then $e_s(\ell_p^n) \leq c_{p,s} n^{(2ps+2s)/(2p-s)}$ for a constant $c_{p,s}$ depending on p and s.

- If the ratio between the largest and smallest distance is bounded, we use an approximation argument with Jackson's Theorem.

Our Results

We can also generalise Alon and Pudlák's theorem to s-distance sets.

Theorem (Chen et. al., 2020)

If s is a positive integer and p is a real number satisfying $2p > s$, then $e_s(\ell_p^n) \leq c_{p,s} n^{(2ps+2s)/(2p-s)}$ for a constant $c_{p,s}$ depending on p and s.

- If the ratio between the largest and smallest distance is bounded, we use an approximation argument with Jackson's Theorem.
- If this ratio is unbounded, our set will look like "clusters" of points. We can use this to induct downwards.

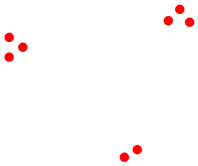
Our Results

We can also generalise Alon and Pudlák's theorem to s-distance sets.

Theorem (Chen et. al., 2020)

If s is a positive integer and p is a real number satisfying $2p > s$, then $e_s(\ell_p^n) \leq c_{p,s} n^{(2ps+2s)/(2p-s)}$ for a constant $c_{p,s}$ depending on p and s.

- If the ratio between the largest and smallest distance is bounded, we use an approximation argument with Jackson's Theorem.
- If this ratio is unbounded, our set will look like "clusters" of points. We can use this to induct downwards.



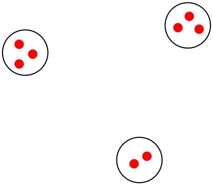
Our Results

We can also generalise Alon and Pudlák's theorem to s-distance sets.

Theorem (Chen et. al., 2020)

If s is a positive integer and p is a real number satisfying $2p > s$, then $e_s(\ell_p^n) \leq c_{p,s} n^{(2ps+2s)/(2p-s)}$ for a constant $c_{p,s}$ depending on p and s.

- If the ratio between the largest and smallest distance is bounded, we use an approximation argument with Jackson's Theorem.
- If this ratio is unbounded, our set will look like "clusters" of points. We can use this to induct downwards.



Our Results

We also use approximation techniques to bound the size of an equilateral set in an ℓ_p sum.

Theorem (Chen et. al., 2020)

Let $\mathbb{E}^{a_1}, \ldots, \mathbb{E}^{a_n}$ be Euclidean spaces and set $a = \max_{1 \le i \le n} a_i$. If $2p > a$, then $e(\mathbb{E}^{a_1} \oplus_p \cdots \oplus_p \mathbb{E}^{a_n}) \le c_{p,a} n^{\frac{2p+2a}{2p-a}}$ for a constant $c_{p,a}$ depending on p and a.

Acknowledgements

We would like to thank:

- Our mentor Feng Gui
- The MIT PRIMES program
- Prof. Larry Guth for proposing the problem
- Prof. Yufei Zhao for helpful suggestions
- Our parents

References

N. Alon and P. Pudlák.
Equilateral sets in l_p^n.
Geom. Funct. Anal., 13(3):467–482, 2003.

Aart Blokhuis.
Few-Distance Sets.
PhD thesis, Technische Universiteit Eindhoven, 1983.

Richard K. Guy.
An olla-podrida of open problems, often oddly posed.
Amer. Math. Monthly, 90(3):196–200, 1983.

Clifford Smyth.
Equilateral sets in ℓ_p^d.
In János Pach, editor, *Thirty Essays on Geometric Graph Theory*, pages 483–487. Springer-Verlag New York, 2013.