On Quasisymmetric Functions with Two Bordering Variables

Alexander Zhang
Mentor: Andrey Khesin

October 17-18, 2020
MIT PRIMES Conference

Introductory Definitions

- The **extended natural numbers**: the set $\mathbb{N} \cup \{0, \infty\}$, denoted by $\mathcal{N}$. Its total order is given by $0 \prec 1 \prec 2 \prec \cdots \prec \infty$.

Introductory Definitions

- The **extended natural numbers**: the set $\mathbb{N} \cup \{0, \infty\}$, denoted by $\mathcal{N}$. Its total order is given by $0 \prec 1 \prec 2 \prec \cdots \prec \infty$.
- $\mathbb{Z}[[x_0, x_1, x_2, \ldots, x_\infty]]$: Ring of formal power series in the variables $x_0, x_1, x_2, \ldots, x_\infty$ over $\mathbb{Z}$. Essentially the ring of polynomials except elements may contain infinitely many monomials.

Introductory Definitions

- The **extended natural numbers**: the set $\mathbb{N} \cup \{0, \infty\}$, denoted by $\mathcal{N}$. Its total order is given by $0 \prec 1 \prec 2 \prec \cdots \prec \infty$.
- $\mathbb{Z}[[x_0, x_1, x_2, \ldots, x_\infty]]$: Ring of formal power series in the variables $x_0, x_1, x_2, \ldots, x_\infty$ over $\mathbb{Z}$. Essentially the ring of polynomials except elements may contain infinitely many monomials.
  - Example: $x_0 + x_1 + x_2 + \cdots + x_\infty = \sum_{i \in \mathcal{N}} x_i$.

Introductory Definitions

- The **extended natural numbers**: the set $\mathbb{N} \cup \{0, \infty\}$, denoted by $\mathcal{N}$. Its total order is given by $0 \prec 1 \prec 2 \prec \cdots \prec \infty$.
- $\mathbb{Z}[[x_0, x_1, x_2, \ldots, x_\infty]]$: Ring of formal power series in the variables $x_0, x_1, x_2, \ldots, x_\infty$ over $\mathbb{Z}$. Essentially the ring of polynomials except elements may contain infinitely many monomials.
  - Example: $x_0 + x_1 + x_2 + \cdots + x_\infty = \sum_{i \in \mathcal{N}} x_i$.
- x_0 and x_∞ are **bordering variables**; x_i for all $i \in \mathbb{N}$ are **natural variables**.

Introductory Definitions

- The **extended natural numbers**: the set $\mathbb{N} \cup \{0, \infty\}$, denoted by $\mathcal{N}$. Its total order is given by $0 \prec 1 \prec 2 \prec \cdots \prec \infty$.
- $\mathbb{Z}[[x_0, x_1, x_2, \ldots, x_\infty]]$: Ring of formal power series in the variables $x_0, x_1, x_2, \ldots, x_\infty$ over $\mathbb{Z}$. Essentially the ring of polynomials except elements may contain infinitely many monomials.
  - Example: $x_0 + x_1 + x_2 + \cdots + x_\infty = \sum_{i \in \mathcal{N}} x_i$.
- x_0 and x_∞ are **bordering variables**; x_i for all $i \in \mathbb{N}$ are **natural variables**.
- Denote by $[m](p)$ the coefficient of the monomial m in the power series p.

Introductory Definitions

- The **extended natural numbers**: the set $\mathbb{N} \cup \{0, \infty\}$, denoted by $\mathcal{N}$. Its total order is given by $0 \prec 1 \prec 2 \prec \cdots \prec \infty$.
- $\mathbb{Z}[[x_0, x_1, x_2, \ldots, x_\infty]]$: Ring of formal power series in the variables $x_0, x_1, x_2, \ldots, x_\infty$ over $\mathbb{Z}$. Essentially the ring of polynomials except elements may contain infinitely many monomials.
  - Example: $x_0 + x_1 + x_2 + \cdots + x_\infty = \sum_{i \in \mathcal{N}} x_i$.
- x_0 and x_∞ are **bordering variables**; x_i for all $i \in \mathbb{N}$ are **natural variables**.
- Denote by $[m](p)$ the coefficient of the monomial m in the power series p.
  - Example: $[x^3 y](x + 7y + 13x^3 y) = 13$.

Quasisymmetric Functions

The **ring of quasisymmetric functions** (QSym): the ring of power series f in the natural variables $x_1, x_2, \ldots$ such that

$$[x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}](f) = [x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}](f)$$

for all $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.

Quasisymmetric Functions

The **ring of quasisymmetric functions** (QSym): the ring of power series f in the natural variables $x_1, x_2, \ldots$ such that

$$[x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}](f) = [x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}](f)$$

for all $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.

Example of a quasisymmetric function: $\sum_{i<j<k} 6x_i x_j^2 x_k^3 + \sum_{i<j} x_i x_j$.

Quasisymmetric Functions

The **ring of quasisymmetric functions** (QSym): the ring of power series f in the natural variables $x_1, x_2, \ldots$ such that

$$[x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}](f) = [x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}](f)$$

for all $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.

Example of a quasisymmetric function: $\sum_{i<j<k} 6x_i x_j^2 x_k^3 + \sum_{i<j} x_i x_j$.

Example of a non-quasisymmetric function:

$$f = x_1 x_2^2 x_3^3 + x_1 x_2^2 x_4^3 + x_1 x_3^2 x_4^3 + x_2 x_3^2 x_4^3$$

because $[x_1 x_2^2 x_3^3](f) \neq [x_1 x_2^2 x_5^3](f)$.

Quasisymmetric Functions

The **ring of quasisymmetric functions** (QSym): the ring of power series f in the natural variables $x_1, x_2, \ldots$ such that

$$[x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}](f) = [x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}](f)$$

for all $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.

Example of a quasisymmetric function: $\sum_{i<j<k} 6x_i x_j^2 x_k^3 + \sum_{i<j} x_i x_j$.

Example of a non-quasisymmetric function:

$$f = x_1 x_2^2 x_3^3 + x_1 x_2^2 x_4^3 + x_1 x_3^2 x_4^3 + x_2 x_3^2 x_4^3$$

because $[x_1 x_2^2 x_3^3](f) \neq [x_1 x_2^2 x_5^3](f)$.

We will investigate power series quasisymmetric in $x_1, x_2, \ldots$ but with two bordering variables x_0, x_∞.

Our Family of Power Series

Let $n \in \mathbb{N} \cup \{0\}$, $[n] = \{1, 2, \ldots, n\}$, and $\Lambda \subseteq [n]$. Define $K_{n,\Lambda}$ as:

Our Family of Power Series

Let $n \in \mathbb{N} \cup \{0\}$, $[n] = \{1, 2, \ldots, n\}$, and $\Lambda \subseteq [n]$. Define $K_{n,\Lambda}$ as:

$$K_{n,\Lambda} = \sum_{\substack{(g_1, g_2, \ldots, g_n) \in \mathcal{N}^n; \\ 0 \preceq g_1 \preceq g_2 \preceq \cdots \preceq g_n \preceq \infty; \\ \text{no } i \in \Lambda \text{ satisfies } g_{i-1} = g_i = g_{i+1} \\ (\text{where } g_0 = 0 \text{ and } g_{n+1} = \infty)}} 2^{|\{g_1, g_2, \ldots, g_n\} \cap \mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}.$$

Our Family of Power Series

Let $n \in \mathbb{N} \cup \{0\}$, $[n] = \{1, 2, \ldots, n\}$, and $\Lambda \subseteq [n]$. Define $K_{n,\Lambda}$ as:

$$K_{n,\Lambda} = \sum_{\substack{(g_1,g_2,\ldots,g_n)\in\mathcal{N}^n; \\ 0\preceq g_1\preceq g_2\preceq\cdots\preceq g_n\preceq\infty; \\ \text{no } i\in\Lambda \text{ satisfies } g_{i-1}=g_i=g_{i+1} \\ (\text{where } g_0=0 \text{ and } g_{n+1}=\infty)}} 2^{|\{g_1,g_2,\ldots,g_n\}\cap\mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}.$$

The sum runs over all combinations with replacement of n nondecreasing elements $g_1, g_2, \ldots, g_n \in \mathcal{N}$ such that for all $i \in \Lambda$, $g_{i-1} = g_i = g_{i+1}$ is false, where $g_0 = 0$ and $g_{n+1} = \infty$.

Our Family of Power Series

Let $n \in \mathbb{N} \cup \{0\}$, $[n] = \{1, 2, \ldots, n\}$, and $\Lambda \subseteq [n]$. Define $K_{n,\Lambda}$ as:

$$K_{n,\Lambda} = \sum_{\substack{(g_1,g_2,\ldots,g_n)\in\mathcal{N}^n; \\ 0\preceq g_1\preceq g_2\preceq\cdots\preceq g_n\preceq\infty; \\ \text{no } i\in\Lambda \text{ satisfies } g_{i-1}=g_i=g_{i+1} \\ (\text{where } g_0=0 \text{ and } g_{n+1}=\infty)}} 2^{|\{g_1,g_2,\ldots,g_n\}\cap\mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}.$$

The sum runs over all combinations with replacement of n nondecreasing elements $g_1, g_2, \ldots, g_n \in \mathcal{N}$ such that for all $i \in \Lambda$, $g_{i-1} = g_i = g_{i+1}$ is false, where $g_0 = 0$ and $g_{n+1} = \infty$.

Our Family of Power Series

Let $n \in \mathbb{N} \cup \{0\}$, $[n] = \{1, 2, \ldots, n\}$, and $\Lambda \subseteq [n]$. Define $K_{n,\Lambda}$ as:

$$K_{n,\Lambda} = \sum_{\substack{(g_1,g_2,\ldots,g_n)\in\mathcal{N}^n; \\ 0\preceq g_1\preceq g_2\preceq\cdots\preceq g_n\preceq\infty; \\ \text{no } i\in\Lambda \text{ satisfies } g_{i-1}=g_i=g_{i+1} \\ (\text{where } g_0=0 \text{ and } g_{n+1}=\infty)}} 2^{|\{g_1,g_2,\ldots,g_n\}\cap\mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}.$$

The sum runs over all combinations with replacement of n nondecreasing elements $g_1, g_2, \ldots, g_n \in \mathcal{N}$ such that for all $i \in \Lambda$, $g_{i-1} = g_i = g_{i+1}$ is false, where $g_0 = 0$ and $g_{n+1} = \infty$.

Example: If $n = 5$ and $\Lambda = \{1, 4\}$, then the combinations $(0, 0, 1, 3, \infty)$ and $(2, 2, 7, 7, 7)$ are not included because they have $g_0 = g_1 = g_2$ and $g_3 = g_4 = g_5$, but $(0, 1, 5, 5, 8)$ is included.

Our Family of Power Series (Continued)

Let $n \in \mathbb{N} \cup \{0\}$, $[n] = \{1, 2, \ldots, n\}$, and $\Lambda \subseteq [n]$. Define $K_{n,\Lambda}$ as:

$$K_{n,\Lambda} = \sum_{\substack{(g_1,g_2,\ldots,g_n)\in\mathcal{N}^n;\\ 0\preceq g_1\preceq g_2\preceq\cdots\preceq g_n\preceq\infty;\\ \text{no } i\in\Lambda \text{ satisfies } g_{i-1}=g_i=g_{i+1}\\ (\text{where } g_0=0 \text{ and } g_{n+1}=\infty)}} 2^{|\{g_1,g_2,\ldots,g_n\}\cap\mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}.$$

Each summand of $K_{n,\Lambda}$ is the monomial $x_{g_1} x_{g_2} \cdots x_{g_n}$ multiplied by 2 to the power of the number of distinct natural variables in $\{x_{g_1}, x_{g_2}, \ldots, x_{g_n}\}$.

Our Family of Power Series (Continued)

Let $n \in \mathbb{N} \cup \{0\}$, $[n] = \{1, 2, \ldots, n\}$, and $\Lambda \subseteq [n]$. Define $K_{n,\Lambda}$ as:

$$K_{n,\Lambda} = \sum_{\substack{(g_1,g_2,\ldots,g_n) \in \mathcal{N}^n; \\ 0 \preceq g_1 \preceq g_2 \preceq \cdots \preceq g_n \preceq \infty; \\ \text{no } i \in \Lambda \text{ satisfies } g_{i-1} = g_i = g_{i+1} \\ (\text{where } g_0 = 0 \text{ and } g_{n+1} = \infty)}} 2^{|\{g_1, g_2, \ldots, g_n\} \cap \mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}.$$

Each summand of $K_{n,\Lambda}$ is the monomial $x_{g_1} x_{g_2} \cdots x_{g_n}$ multiplied by 2 to the power of the number of distinct natural variables in $\{x_{g_1}, x_{g_2}, \ldots, x_{g_n}\}$.

Example: If $n = 5$ and $(g_1, g_2, \ldots, g_n) = (0, 1, 5, 5, 7)$, then the corresponding summand $2^{|\{g_1, g_2, \ldots, g_n\} \cap \mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}$ equals $8 x_0 x_1 x_5^2 x_7$.

Examples of $K_{n,\Lambda}$'s

$$K_{n,\Lambda} = \sum_{\substack{(g_1,g_2,\ldots,g_n)\in\mathcal{N}^n; \\ 0\preceq g_1\preceq g_2\preceq\cdots\preceq g_n\preceq\infty; \\ \text{no } i\in\Lambda \text{ satisfies } g_{i-1}=g_i=g_{i+1} \\ (\text{where } g_0=0 \text{ and } g_{n+1}=\infty)}} 2^{|\{g_1,g_2,\ldots,g_n\}\cap\mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}.$$

Examples of $K_{n,\Lambda}$'s

$$K_{n,\Lambda} = \sum_{\substack{(g_1,g_2,\ldots,g_n)\in\mathcal{N}^n;\\ 0\preceq g_1\preceq g_2\preceq\cdots\preceq g_n\preceq\infty;\\ \text{no } i\in\Lambda \text{ satisfies } g_{i-1}=g_i=g_{i+1}\\ (\text{where } g_0=0 \text{ and } g_{n+1}=\infty)}} 2^{|\{g_1,g_2,\ldots,g_n\}\cap\mathbb{N}|} x_{g_1}x_{g_2}\cdots x_{g_n}.$$

- When $n = 1$, $K_{1,\{\}} = K_{1,\{1\}} = x_0 + \sum_{i\in\mathbb{N}} 2x_i + x_\infty$. There are no restrictions on g_1 because $0 = g_1 = \infty$ is impossible.

Examples of $K_{n,\Lambda}$'s

$$K_{n,\Lambda} = \sum_{\substack{(g_1,g_2,\dots,g_n)\in\mathcal{N}^n;\\ 0\preceq g_1\preceq g_2\preceq\cdots\preceq g_n\preceq\infty;\\ \text{no } i\in\Lambda \text{ satisfies } g_{i-1}=g_i=g_{i+1}\\ (\text{where } g_0=0 \text{ and } g_{n+1}=\infty)}} 2^{|\{g_1,g_2,\dots,g_n\}\cap\mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}.$$

- When $n = 1$, $K_{1,\{\}} = K_{1,\{1\}} = x_0 + \sum_{i\in\mathbb{N}} 2x_i + x_\infty$. There are no restrictions on g_1 because $0 = g_1 = \infty$ is impossible.
- When $n = 2$,

$$K_{2,\{1\}} = \sum_{i\in\mathbb{N}} 2x_0x_i + \sum_{i\in\mathbb{N}} 2x_ix_\infty + \sum_{i\in\mathbb{N}} 2x_i^2 + \sum_{i<j\in\mathbb{N}} 4x_ix_j + x_\infty^2.$$

The coefficient of each monomial depends on how many distinct natural variables it contains.

Main Theorem

Theorem

The span of our family of power series, $\operatorname{span}\left(K_{n,\Lambda}\right)_{n\in\mathbb{N}\cup\{0\};\ \Lambda\subseteq[n]}$, *is a* $\mathbb{Z}$*-subalgebra of* $\mathbb{Z}[[x_0, x_1, x_2, \ldots, x_\infty]]$.

Main Theorem

Theorem

The span of our family of power series, $\text{span}\,(K_{n,\Lambda})_{n\in\mathbb{N}\cup\{0\};\ \Lambda\subseteq[n]}$*, is a* $\mathbb{Z}$*-subalgebra of* $\mathbb{Z}[[x_0, x_1, x_2, \ldots, x_\infty]]$*.*

Equivalently, the product of any $K_{n,\Lambda}K_{m,\Omega}$ can be written as the sums and differences of several $K_{n+m,\Xi}$'s, where each $\Xi \subseteq [n+m]$.

Main Theorem

Theorem

The span of our family of power series, span $(K_{n,\Lambda})_{n\in\mathbb{N}\cup\{0\};\ \Lambda\subseteq[n]}$, *is a* $\mathbb{Z}$*-subalgebra of* $\mathbb{Z}[[x_0, x_1, x_2, \ldots, x_\infty]]$.

Equivalently, the product of any $K_{n,\Lambda}K_{m,\Omega}$ can be written as the sums and differences of several $K_{n+m,\Xi}$'s, where each $\Xi \subseteq [n+m]$.

Example: When $n = m = 2$ and $\Lambda = \Omega = \{1, 2\}$, the product $K_{2,\{1,2\}}K_{2,\{1,2\}}$ can be written as

$$K_{4,\{2\}} + 2K_{4,\{1,3\}} + 2K_{4,\{1,4\}} + K_{4,\{2,4\}} + K_{4,\{1,2,4\}} - K_{4,\{1,2\}}.$$

Origin of the Problem

Grinberg proved the theorem in 2018 for exterior peak sets Λ, Ω; i.e. sets with no pair of consecutive integers. That result was key to proving the shuffle-compatibility of the exterior peak set statistic Epk.

Origin of the Problem

Grinberg proved the theorem in 2018 for exterior peak sets Λ, Ω; i.e. sets with no pair of consecutive integers. That result was key to proving the shuffle-compatibility of the exterior peak set statistic Epk.

In 2020, Grinberg later proved the theorem for $x_0 = x_\infty = 0$, in which case the power series are quasisymmetric.

Proof Outline

Consider similar family of functions, $L_{n,\Lambda}$'s, which can be handled more easily; prove the following equivalent theorem: the product of any $L_{n,\Lambda}L_{m,\Omega}$ can be written as the sums and differences of several $L_{n+m,\Xi}$'s, where each $\Xi \subseteq [n+m]$.

Proof Outline

Consider similar family of functions, $L_{n,\Lambda}$'s, which can be handled more easily; prove the following equivalent theorem: the product of any $L_{n,\Lambda}L_{m,\Omega}$ can be written as the sums and differences of several $L_{n+m,\Xi}$'s, where each $\Xi \subseteq [n+m]$.

"Zero out" the coefficients of monomials in $L_{n,\Lambda}L_{m,\Omega}$ by adding/subtracting $L_{n+m,\Xi}$'s in a specific order.

Proof Outline

Consider similar family of functions, $L_{n,\Lambda}$'s, which can be handled more easily; prove the following equivalent theorem: the product of any $L_{n,\Lambda}L_{m,\Omega}$ can be written as the sums and differences of several $L_{n+m,\Xi}$'s, where each $\Xi \subseteq [n+m]$.

"Zero out" the coefficients of monomials in $L_{n,\Lambda}L_{m,\Omega}$ by adding/subtracting $L_{n+m,\Xi}$'s in a specific order.

Show that doing so results in the coefficients of every monomial becoming zero, thus proving the theorem.

References

D. Grinberg, Shuffle-compatible permutation statistics II: the exterior peak set. *Electron. J. Combin.* **25** (2018), Paper 17.

D. Grinberg, The eta-basis of QSym, 2020. Available at `www.cip.ifi.lmu.de/~grinberg/algebra/etabasis.pdf`.

Acknowledgements

- My mentor, Andrey Khesin
- Professor Darij Grinberg
- The MIT PRIMES program
- My aunt, for letting me use her WiFi