Product Expansions of q-Character Polynomials

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Background: Example

Let A be a nonzero $n \times n$ matrix with entries in the finite field $\mathbb{F}_q$.

Definition

Let $\mathrm{Fix}(A) = |\{v \in \mathbb{F}_q^n \mid Av = v \text{ and } v \neq 0\}|$ be a statistic on matrices of any size.

Question

What is $\mathbb{E}_A[\mathrm{Fix}(A)]$?

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Expectation of Fix(A)

If A has entries in $\mathbb{F}_q$, then $\mathbb{E}_A[\text{Fix}(A)] = 1$.

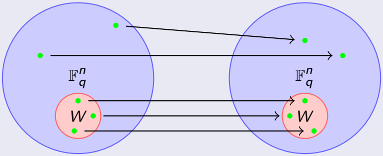
We can extend this notion to counting subspaces rather than vectors.

The Statistic X_B

In this project, we look at the infinite collection of class function X_B, which is defined as follows:

Definition

Given a linear transformation A and vector space $\mathbb{F}_q^n$, we say that a subspace $W \subseteq \mathbb{F}_q^n$ is **A-invariant** if $A(W) \subseteq W$.



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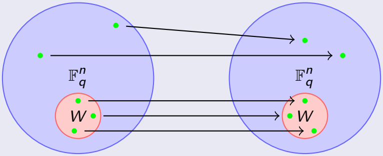

$$A : \mathbb{F}_q^n \to \mathbb{F}_q^n$$

Definition (Conjugacy)

The matrices A and B are **conjugate** if there exists an invertible matrix P such that $B = P^{-1}AP$. Conjugacy is an equivalence relation.

The Statistic X_B (A Generalization)

- v (vector)
- A acts like Id on v
- $\mathrm{Fix}(A)$

- W (subspace)
- A acts like B on W
- $X_B(A)$

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Definition (q-character polynomials)

Given a finite field $\mathbb{F}_q$, let B be an $m \times m$ matrix, where $m \geq 1$. If A is any $n \times n$ matrix,

$$X_B(A) = |\{W \leq \mathbb{F}_q^n \mid \dim W = m \text{ with } A(W) \subseteq W \text{ and } A|_W \sim B\}|.$$

X_B In Action

Notation

Let the $n \times n$ identity matrix be I_n.

Example

$X_{I_1}\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right) = ?$

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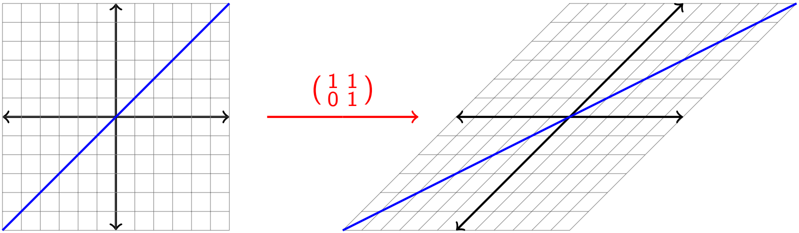

- $X_{I_1}((\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})) = 1$ line.

Motivation

Why is this interesting?

- Distribution of eigenvalues
- Calculating correlation between Jordan blocks
- Studying generalizations of Fix(A)

Research Problem

Dr. Gadish, our mentor, proved that X_B **formed a ring under multiplication**.

Theorem

Given matrices B_1 and B_2 of size k_1 and k_2 respectively, there exists the following expansion for the pointwise product $X_{B_1} \cdot X_{B_2}$.

$$X_{B_1} \cdot X_{B_2} = \sum_C \lambda^C_{B_1,B_2} X_C$$

for some scalars $\lambda^C_{B_1,B_2}$ where the sum ranges over conjugacy classes of invertible matrices C of size $\max(k_1, k_2) \leq k \leq k_1 + k_2$.

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Goal

Characterize the scalars $\lambda^C_{B_1,B_2}$

Identity Matrices

Definition

The q-**binomial coefficient** $\binom{n}{k}_q$ is the number of k-dimensional subspaces in a n-dimensional space.

Fact

$$\binom{n}{k}_q = \prod_{i=0}^{k-1} \frac{q^n - q^i}{q^k - q^i}.$$

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Theorem

The following evaluation is true for identity matrices:

$$X_{I_k}(I_n) = \binom{n}{k}_q.$$

Product of Statistics of Identity Matrices

Theorem (Product of Identity Matrices)

$$X_{I_n} \cdot X_{I_m} = \sum_{k=0}^{\min(m,n)} X_{I_{m+n-k}} \binom{m+n-k}{k}_q \binom{m+n-2k}{m-k}_q \cdot q^{(m-k)(n-k)}.$$

Proof.

- Pick two subspaces V and W of dimension m and n respectively.

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- Consider $V + W$ and $V \cap W$. Let $\dim(V \cap W) = k$.

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Proof.

- Pick two subspaces V and W of dimension m and n respectively.
- Consider $V + W$ and $V \cap W$. Let $\dim(V \cap W) = k$.
- $X_{I_{m+n-k}}$ ways to pick $V + W$.
- $\binom{m+n-k}{k}_q$ ways to pick $V \cap W$.

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Proof.

- Pick two subspaces V and W of dimension m and n respectively.
- Consider $V+W$ and $V \cap W$. Let $\dim(V \cap W) = k$.
- $X_{I_{m+n-k}}$ ways to pick $V+W$.
- $\binom{m+n-k}{k}_q$ ways to pick $V \cap W$.
- $\binom{m+n-2k}{m-k}_q \cdot q^{(m-k)(n-k)}$ ways to pick extensions of $V \cap W$ to make V and W.

□

Reductions: Field Extensions

The product of statistics associated with identity matrices is complicated. The general formula even more so. We can reduce the problem with reductions.

Example (Field Extensions)

Consider the following matrix:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

No real eigenvalues.

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Example (Field Extensions)

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$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

No real eigenvalues. Over the complex numbers, however, it does have eigenvalues:

$$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

Reduction

It is sufficient to assume that the matrix is in Jordan form.

Reductions

Reduction (Disjoint Eigenvalues)

If A and B have disjoint eigenvalues then $X_A \cdot X_B = X_{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}}$.

Reductions

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If A and B have disjoint eigenvalues then $X_A \cdot X_B = X_{\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)}$.

Reduction (Changing Eigenvalues)

The choice of eigenvalue does not matter. For example,

$$X_{\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)}\left(\left(\begin{smallmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{smallmatrix}\right)\right) = X_{\left(\begin{smallmatrix} \mu & 1 \\ 0 & \mu \end{smallmatrix}\right)}\left(\left(\begin{smallmatrix} \mu & 1 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{smallmatrix}\right)\right).$$

Final Reduction

It suffices to calculate our product expansion coefficients with the assumption that the matrices are unipotent.

Procedure for Calculating Coefficients

Example

There exists an expansion $X_{(1)} \times X_{(1)} = aX_{(1)} + bX_{\binom{10}{01}} + cX_{\binom{11}{01}}$.

- Plug in (1), we see that $X_{(1)}((1)) = 1$, so

$$1 \times 1 = 1 \times a + \cancel{b \times 0} + \cancel{c \times 0}.$$

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- Plug in $\binom{10}{01}$, we see that $X_{\binom{10}{01}}((1)) = q + 1$, $X_{\binom{10}{01}}(\binom{10}{01}) = 1$.

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  Therefore, $b = q(q + 1)$.

- Plugging in $\binom{11}{01}$, we see that $c = 0$.

So, $\boxed{X_{(1)} \times X_{(1)} = X_{(1)} + q(q + 1)X_{\binom{10}{01}}}$.

Procedure for Calculating Coefficients

Algorithm to Calculate the Coefficients λ_{B_1,B_2}^{C}

1: **for** $k = \max\{\dim(B_1), \dim(B_2)\}$ to $\dim(B_1) + \dim(B_2)$ **do**
2: Choose a conjugacy class C of k-dimensional matrices
3: Determine $X_{B_1}(C)$, $X_{B_2}(C)$, and $X_M(C)$ where $\dim(M) < \dim(C)$
4: Set $\lambda_{B_1,B_2}^{C} = X_{B_1}(C) \cdot X_{B_2}(C) - \sum_{\dim(M)<\dim(C)} \lambda_{B_1,B_2}^{M} X_M(C)$
5: Repeat with all other conjugacy classes of matrices of dimension k
6: **end for loop**

Evaluating $X_{J_k}(A)$

Notation

$$J_{a_1,a_2,\ldots,a_n} = J_{a_1}(1) \oplus J_{a_2}(1) \oplus \cdots \oplus J_{a_n}(1)$$

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Theorem

$$X_{J_k}(J_{a_1,a_2,\ldots,a_r,b_1,b_2,\ldots,b_s}) = \binom{r}{1}_q q^{\sum_i b_i + (k-1)(r-1)}$$

where $a_i \geq k$ *and* $b_i < k$.

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Proof (sketch).

- Count k-dim subspaces, A-invariant and A acts as a Jordan block of size k.

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Proof (sketch).

- Count k-dim subspaces, A-invariant and A acts as a Jordan block of size k.
- Count vectors that generate such such subspaces, adjust for overcounting

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Observations:

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- Polynomial in q

Main Result: Evaluating $X_B(A)$

Theorem

Let $B = J_{b_1,b_1,\ldots,b_2,\ldots,b_n}$ where $b_1 > b_2 > \cdots > b_n$ and there are c_i copies of J_{b_i} in B. Let $A = J_{a_1,a_2,\ldots,a_k}$ where $a_1 \geq a_2 \geq \cdots \geq a_k$. Let t_i be the largest integer such that $a_{t_i} \geq b_i$. Then,

$$X_B(A) = \left(\prod_{i=1}^{n} \binom{t_i - \sum_{j=1}^{i-1} c_i}{c_i}_q \right) \cdot q^{-\sum_{i<j} c_i c_j + \sum_{i=1}^{n} c_i \left((b_i - 1)(t_i - c_i - 2\sum_{j=1}^{i-1} c_j) + \sum_{j=t_i+1}^{k} a_j \right)}.$$

The proof is analogous.

Statistics of Single Jordan Blocks

Theorem

For $b > a$,

$$X_{J_b} \cdot X_{J_a} = X_{J_b} + q(q-1)X_{J_{b,1}} + \cdots + q^{2a-3}(q-1)X_{J_{b,a-1}} + q^{2a}X_{J_{b,a}}$$
$$X_{J_b}^2 = X_{J_b} + q(q-1)X_{J_{b,1}} + \cdots + q^{2b-3}(q-1)X_{J_{b,b-1}} + q^{2b-1}(q+1)X_{J_{b,b}}$$

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Interesting observations:

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Interesting observations:

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- Expansion coefficients of $X_{J_b} \cdot X_{J_a}$ are stable (independent of b)

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Corollary

X_{J_i} generate all $X_{J_{b,a}}$.

Application of the Expansion of $X_{J_a} \cdot X_{J_b}$

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Formula

$$\text{corr}(X_{J_1}, X_{J_2}) = \frac{\mathbb{E}[X_{J_2} \cdot X_{J_1}] - \mathbb{E}[X_{J_2}]\mathbb{E}[X_{J_1}]}{\sqrt{(\mathbb{E}[X_{J_1}^2] - \mathbb{E}[X_{J_1}]^2)(\mathbb{E}[X_{J_2}^2] - \mathbb{E}[X_{J_2}]^2)}}$$

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- $X_{J_2} \cdot X_{J_1} = X_{J_2} + q^2 X_{J_{2,1}}$
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Result

For all $n \geq 4$, $\mathrm{corr}(X_{J_1}, X_{J_2}) = \frac{\sqrt{q+1}}{q}$.

Conjectures

Conjectures

Conjecture: Expansion for $X_{J_n} \cdot X_{J_{m,m}}$

If $n > m$ are two positive integers,

$$X_{J_n} \cdot X_{J_{m,m}} = q^m X_{J_{n,m}} + \sum_{k=1}^{m-1} q^{3k+m-1}(q-1) X_{J_{n,m,k}} + q^{4m} X_{J_{n,m,m}}.$$

We conjecture similar expansions for $X_{J_n} \cdot X_{J_{m,k}}$.

Conjectures

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Conjecture: Maximum Number of Jordan Blocks

$X_{J_{a_1,a_2,\ldots,a_n}} \cdot X_{J_{b_1,b_2,\ldots,b_m}} = \sum \lambda_C X_{J_{c_1,c_2,\ldots,c_k}}$ where $\max(m,n) \leq k \leq m+n$.

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Conjecture: Single Jordan Blocks Generate the Ring

$X_C \in \mathbb{Q}[X_{J_1}, X_{J_2}, X_{J_3}, \ldots]$ for all C

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- If $GL_n(\mathbb{F}_q) \curvearrowright V, W$, then $V \otimes W$, viewed as a $GL_n(\mathbb{F}_q)$-module, can be decomposed in a direct sum of simple $GL_n(\mathbb{F}_q)$-modules. Our coefficients can be used to determine the multiplicities, which were seen to stabilize.

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