Applications of Homology

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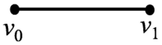

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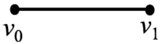

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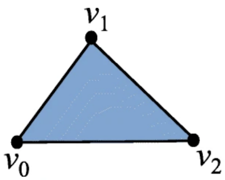

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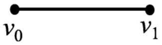

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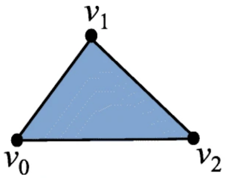

I think you all see where this is headed.

Simplices

Definition (n-Simplex)

A n-simplex in $\mathbb{R}^n$ with vertices $v_0, v_1, \ldots, v_n \in \mathbb{R}^n$, denoted $[v_0, \ldots, v_n]$, is defined to be the set of all points that can be expressed as some weighted average of the vertices v_i. That is,

$$[v_0, \ldots, v_n] = \left\{ \sum_{i=0}^{n} a_i v_i \;\middle|\; \sum_{i=0}^{n} a_i = 1, a_i \geq 0 \text{ for all } i \in [0, n] \right\}.$$

Simplices

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To account for orientation, we must define the following:

Orientation of Simplices

For any two oriented simplices,

$$[v_0, \dots, v_i, \dots, v_j, \dots v_n] = (-1)[v_0, \dots, v_j, \dots, v_i, \dots v_n].$$

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That is, swapping two vertices in the orientation of a simplex negates it.

So in fact, $[v_0, v_1, v_2] = -[v_0, v_2, v_1]$.

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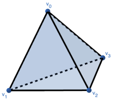

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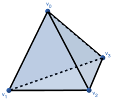
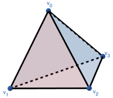


This is a face of $[v_0, v_1, v_2, v_3]$:



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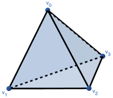

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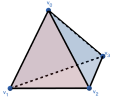

which is represented by $[v_0, v_1, v_2]$.

Similarly, the other faces are $[v_0, v_1, v_3], [v_0, v_2, v_3], [v_1, v_2, v_3]$, where the order of the vertices in each face follow the order in which they are given in the original simplex.

Face

Definition (Face)

Upon deletion of one vertex x_i from a n-simplex $\triangle^n$, the remaining $n-1$ simplex $[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ is a face of $\triangle^n$.

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For example, the 2-simplex $[v_0, v_1, v_2]$ has the faces $[v_1, v_2], [v_0, v_2], [v_0, v_1]$, which are formed by removing each of v_0, v_1, and v_2 from $[v_0, v_1, v_2]$.

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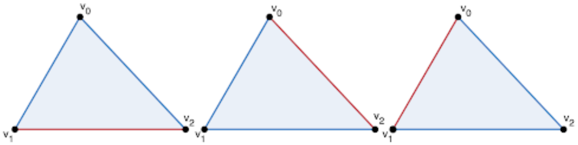

Simplicial Complexes

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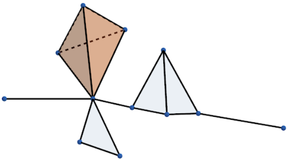

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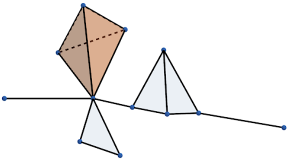

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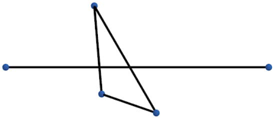

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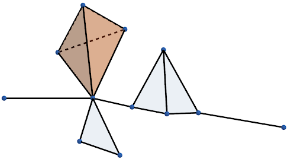

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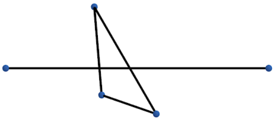


So after all, there have to be rules governing how the simplices can be put together.

Simplicial Complexes

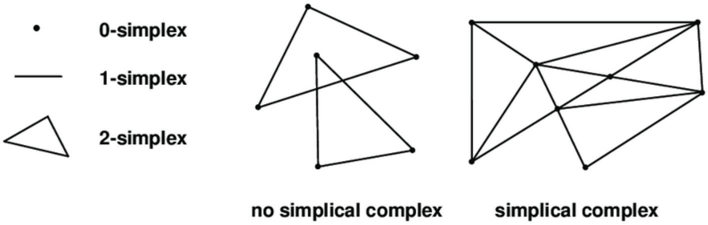
Definition (Simplicial Complex)

A simplicial complex $\mathcal{K}$ is a set of simplices satisfying the following properties:

- Any face of any simplex in $\mathcal{K}$ is in $\mathcal{K}$
- The intersection of any two simplices in $\mathcal{K}$ must either be empty or a face of both simplices.

Simplicial Complexes

Another example for what is and is not a simplicial complex:



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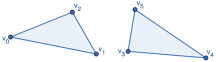

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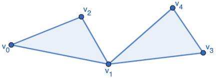

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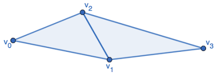

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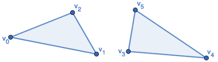

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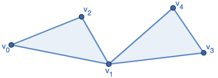

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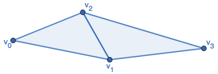

Note that multiples of a simplex are also defined, that is, for a simplex ρ, the term $n\rho$ simply represents n copies of ρ overlaid on top of each other.

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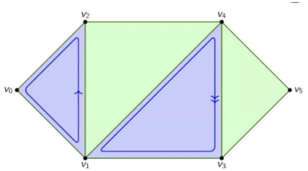
For example, in the simplicial complex shown below,



we have highlighted the n-chain $[v_0, v_1, v_2] - 2[v_1, v_3, v_4]$ in blue.

Simplicial Complexes

Definition (Chain Complexes)

Given a simplicial complex X, let $C_n(X)$ be the free abelian group with its elements as all the n-chains of X. More formally,

$$C_n(X) = \left\{ \sum_{\triangle_i^n \in S} a_i \triangle_i^n \mid a_i \in \mathbb{Z} \right\}$$

where S is the set of n-simplices of X.

Definition (Boundary)

The boundary $\partial\triangle^n$ of a n-simplex $\triangle^n = [v_0, v_1, \ldots, v_n]$ is defined to be

$$\begin{aligned}\partial\triangle^n &= \sum_{i=0}^{n}(-1)^i[v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]\\ &= [v_1, v_2, \ldots, v_n] - [v_0, v_2, \ldots, v_n] + \cdots + (-1)^n[v_0, v_1, \ldots, v_{n-1}].\end{aligned}$$

where signs exist to make sure all vertices of the simplices are oriented correctly in the boundary.

Simplicial Complexes and Homology

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where signs exist to make sure all vertices of the simplices are oriented correctly in the boundary.

This is a little tough to unpack at first glance, but this is really just saying that the boundary of a simplex ρ is an alternating sum of all of its faces.

Simplicial Complexes

Let's take a look at some examples:

For example, intuitively, the boundary of $[a, b, c]$ if we start at a and go in the counterclockwise orientation is

$$\partial[a, b, c] = [a, b] + [b, c] + [c, a] = [b, c] - [a, c] + [a, b],$$

indeed satisfying the definition. We see that the purpose of the negative sign here is indeed to preserve the counterclockwise orientation.

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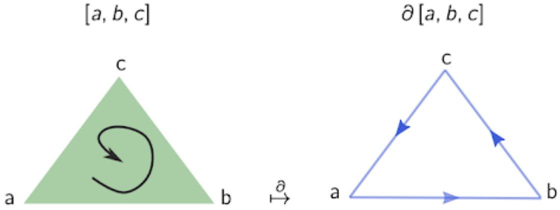

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Having defined the boundary operator for simplices, we are able to define the boundary operator for any linear combination of simplices due to the additivity of ∂. This leads to:

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Definition (Boundary Map)

The n-th boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ on a simplicial complex X is a homomorphism defined by

$$\partial_n(\rho) = \partial(\rho) \text{ for every } \rho \in C_n(X).$$

Simplicial Complexes

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Let's head back to that example with $[a, b, c]$. What happpens if we take the boundary again? We get

$$\begin{aligned}\partial^2[a, b, c] &= \partial[b, c] - \partial[a, c] + \partial[a, b] \\ &= ([c] - [b]) - ([c] - [a]) + ([b] - [a]) \\ &= 0.\end{aligned}$$

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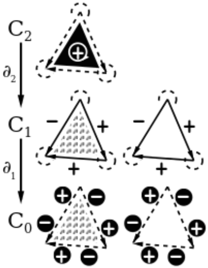
Indeed, it turns out that it applying the boundary operation twice on any simplex sends it to 0.



Lemma ($\partial^2 = 0$)

The composition map $\partial_n \circ \partial_{n+1}$ for any n is the 0 map.

Note that $\partial_n \circ \partial_{n+1} = 0$ implies $\mathrm{Im}(\partial_{n+1}) \subset \mathrm{Ker}(\partial_n)$.

Homology

Definition (Homology Group)

The n-th homology group H_n of a simplicial complex X is defined to be

$$H_n(X) = \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1}).$$

Extensions of Homology

So the reason we started off with simplices and simplicial complexes is that homology is easier with them.

Furthermore, the idea of a simplicial complex is useful because it allows us to reduce hard-to-grasp geometric figures to shapes that are easier to deal with using algebra, which is something we are familiar with.

That being said, homology can actually extended to all sorts of geometric spaces, as long as they can be topologically reduced to simplicial complexes.

Applications of Homology

Theorem

A continuous map $f : X \to Y$ between two geometric spaces X, Y induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ for every n.

Now, we use will prove a famous application of homology.

Brouwer's Fixed Point Theorem

Brouwer's Fixed Point Theorem

Every continuous map $f : D^n \to D^n$ has a fixed point, where D^n is the closed unit disk in $\mathbb{R}^n$.

Brouwer's Fixed Point Theorem

A famous real-life example of Brouwer's Fixed Point Theorem is exhibited by the following scenario:

- Consider a map, and overlay a rotated and dilated copy of it on top of the original (it has to be entirely contained within it). It is always possible to pin a point that marks the same location on both maps.

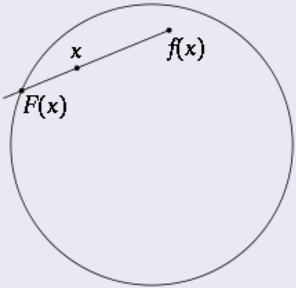
Proof Sketch of Brouwer

Part 1: (Defining $F : D^n \to \partial D^n$)

Instead, suppose f has no fixed point. We define $F : D^n \to \partial D^n$ from f:

- For each $x \in D^n$, let the ray from $f(x)$ to x intersect the boundary of D^n at $F(x)$.

This induces a continuous map F bringing each point in D^n to a point on its boundary, ∂D^n, also known as a retraction.



Proof Sketch of Brouwer

Part 2: Homology Group Map induced by F

$F : D^n \to \partial D^n$ induces a homomorphism

$$\phi_F : H_{n-1}(D^n) \to H_{n-1}(\partial D^n).$$

Let $i : \partial D^n \to D^n$ be the inclusion map, which induces a homomorphism

$$\phi_i : H_{n-1}(\partial D^n) \to H_{n-1}(D^n).$$

Note that $\phi_F \circ \phi_i$ is the identity map on $H_{n-1}(\partial D^n)$, which turns out to be impossible due to $H_{n-1}(D^n) \cong 0$ and $H_{n-1}(\partial D^n) \cong \mathbb{Z}$, contradiction.

Acknowledgements

I would like to thank the following for their continued support:

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- Slava Gerovitch and Pavel Etingof for running the PRIMES program
- Tanya Khovanova and Kent Vashaw for helping me organize this presentation
- my parents
- and finally you all for listening

References

1. Allen Hatcher: Algebraic Topology, https://pi.math.cornell.edu/ hatcher/AT/AT.pdf