A High-order Cumulant-based Sparse Ruler

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The Sparse Ruler

The Sparse Ruler

Definition 1.1. *A sparse ruler is a set of integers* $\mathbb{S} = \{s_1, s_2, \ldots, s_n\}$. *We say that* S *generates the set of lags* $\Phi(\mathbb{S})$ *if for any integer* $\phi \in \Phi(\mathbb{S})$, *there are* i, j *such that* $s_i - s_j = \phi$.

Problem: given a fixed number (n) of marks, how do we construct the ruler (S) to maximize the number of consecutive lags (Φ)?

Motivation

- NP-Completeness
- Information Theory
- Error-Correcting Code
- Signal Processing

Nested Ruler

$$\mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2$$

$$\mathbb{S}_1 = \{n_1 N_2 \mid n_1 = 1, 2, \ldots, N_1\}$$

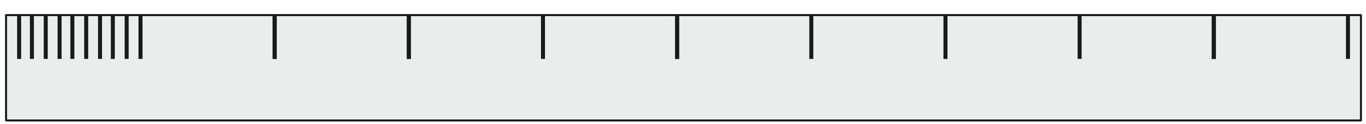
$$\mathbb{S}_2 = \{n_2 \mid n_2 = 1, 2, \ldots, N_2\}$$

$$\text{Then, } \Phi(\mathbb{S}) = \{\mu \mid -N_1 N_2 + 1 \leq \mu \leq N_2 N_2 - 1\}$$

Example: take

$$1, 2, 3, \ldots (10)$$

$$10, 20, 30, \ldots 100$$



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$$\mathbb{S}_1 = \{n_1 N_2 \mid n_1 = 1, 2, \ldots, N_1\}$$

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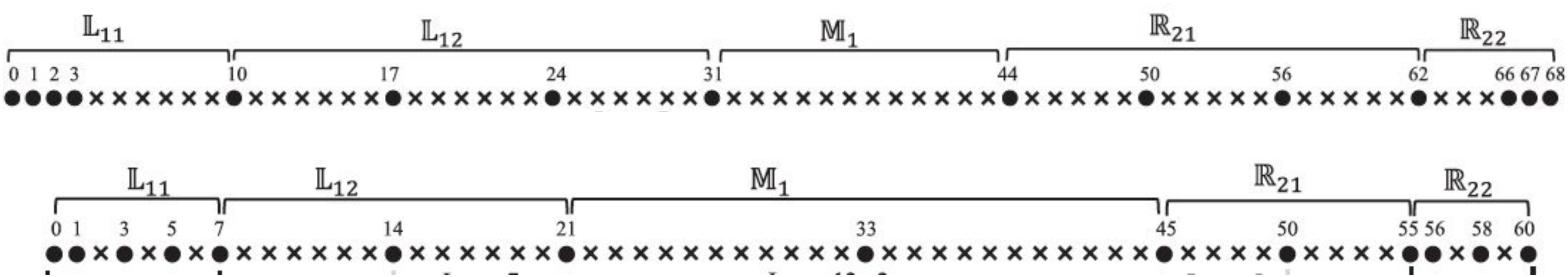
$$\text{Then, } \Phi(\mathbb{S}) = \{\mu \mid -N_1 N_2 + 1 \leq \mu \leq N_2 N_2 - 1\}$$

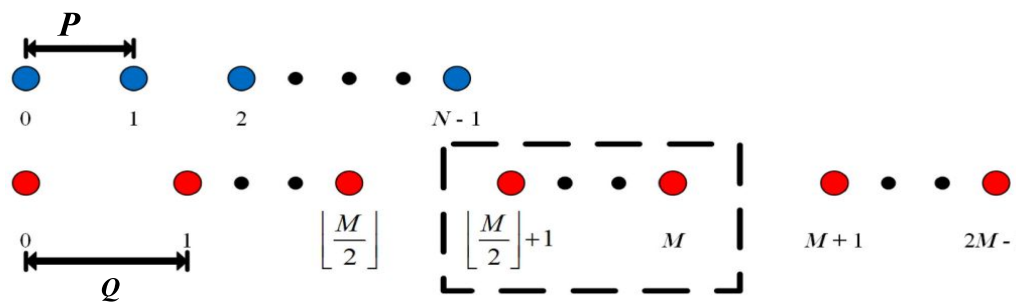
Variations:



Coprime Ruler

General form: For integers P, Q where $\gcd(P, Q) = 1$

$$\mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2$$
$$\mathbb{S}_1 = \{n_1 \cdot P \mid n_1 = 0, 1, 2, \ldots, Q1\}$$
$$\mathbb{S}_2 = \{n_2 \cdot Q \mid n_2 = 0, 1, 2, \ldots, 2P - 1\}$$
$$\text{Then, } \Phi(\mathbb{S}) = \{\mu \mid -PQ - P + 1 \leq \mu \leq PQ + P - 1\}$$



Cumulants and High-Order Rulers

Definition 2.1. *Consider a ruler $\mathbb{S}$. The set of 2q-th order lags*

$$\Phi^{2q}(\mathbb{S}) = \{\sum_{i=1}^{q} p_{n_i} - \sum_{i=q+1}^{2q} p_{n_i} \mid n_i \in [1, N]\}$$

We denote Φ^2 as Φ for short.

Benefit: Increased lag generation from O(N^2) to O(N^2q)

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But... Not trivial:

- 4th-Order: s_a, s_b, s_c, s_d has $\binom{4}{2} = 6$ sign permutations
- $2q$th-Order has $\binom{2q}{q}$

4th-Order: General Form

$$\mathbb{S} = \mathbb{P}_1 \cup \mathbb{P}_2 \cup \mathbb{Q}_1 \cup \mathbb{Q}_2$$

$$\begin{aligned} \text{Examine} \quad : &\pm (p_1 + p_2) - (q_1 + q_2) \\ &\pm (p_1 - p_2) - (q_1 - q_2) \\ &\pm (p_1 + p_2) + (q_1 + q_2) \end{aligned}$$

Nested

Nested

Coprime

4th-Order: General Form

$$\mathbb{S} = \mathbb{P}_1 \cup \mathbb{P}_2 \cup \mathbb{Q}_1 \cup \mathbb{Q}_2$$

$$\mathbb{P}_1 = \{(n_1 N_2 + \left\lfloor \frac{q}{2} \right\rfloor)p \mid n_1 = 0, 1, 2, \ldots, N_1\}$$

$$\mathbb{P}_2 = \{(n_2 + q)p \mid n_2 = 0, 1, 2, \ldots, N_2\}$$

$$\mathbb{Q}_1 = \{(n_3 N_4 - \left\lfloor \frac{p}{2} \right\rfloor)q \mid n_4 = 0, 1, 2, \ldots, N_4\}$$

$$\mathbb{Q}_2 = \{(n_4 - \left\lfloor \frac{p}{2} \right\rfloor)q \mid n_5 = 0, 1, 2, \ldots, N_5\}$$

4th-Order: General Form

$$\mathbb{S} = \mathbb{P}_1 \cup \mathbb{P}_2 \cup \mathbb{Q}_1 \cup \mathbb{Q}_2$$

$$\mathbb{P}_1 = \{(n_1 N_2 + \left\lfloor \frac{q}{2} \right\rfloor)p \mid n_1 = 0, 1, 2, \ldots, N_1\}$$

$$\mathbb{P}_2 = \{(n_2 + q)p \mid n_2 = 0, 1, 2, \ldots, N_2\}$$

- Nested Ruler
- Common multiple of *p*
- Shifted by a factor
  - Lemma: Shifting adds up

4th-Order: Integration

$$\mathbb{S} = \mathbb{P}_1 \cup \mathbb{P}_2 \cup \mathbb{Q}_1 \cup \mathbb{Q}_2$$

$$\mathbb{P}_1 = \{(n_1 N_2 + \left\lfloor \frac{q}{2} \right\rfloor)p \mid n_1 = 0, 1, 2, \ldots, N_1\}$$

$$\mathbb{P}_2 = \{(n_2 + q)p \mid n_2 = 0, 1, 2, \ldots, N_2\}$$

$$\mathbb{Q}_1 = \{(n_3 N_4 - \left\lfloor \frac{p}{2} \right\rfloor)q \mid n_4 = 0, 1, 2, \ldots, N_4\}$$

$$\mathbb{Q}_2 = \{(n_4 - \left\lfloor \frac{p}{2} \right\rfloor)q \mid n_5 = 0, 1, 2, \ldots, N_5\}$$

- $\mathbb{P}$ and $\mathbb{Q}$: larger coprime structure
- Orienting $(p_1 + p_2) - (q_1 + q_2)$, $(p_1 - p_2) - (q_1 + q_2)$, $(p_1 + p_2) + (q_1 + q_2)$

4th-Order: Integration + Result

- Orienting $(p_1+p_2)-(q_1+q_2)$, $(p_1-p_2)-(q_1+q_2)$, $(p_1+p_2)+(q_1+q_2)$

12	17	22	27	32	37	42	47	52	57	62	67	72	77	82	87	92	97	102	107	112	117	122	127	132	137	142	147	152	157	162	167	172	177	182	187
6	11	16	21	26	31	36	41	46	51	56	61	66	71	76	81	86	91	96	101	106	111	116	121	126	131	136	141	146	151	156	161	166	171	176	181
0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125	130	135	140	145	150	155	160	165	170	175
6	1	4	9	14	19	24	29	34	39	44	49	54	59	64	69	74	79	84	89	94	99	104	109	114	119	124	129	134	139	144	149	154	159	164	169
12	7	2	3	8	13	18	23	28	33	38	43	48	53	58	63	68	73	78	83	88	93	98	103	108	113	118	123	128	133	138	143	148	153	158	163
18	13	8	3	2	7	12	17	22	27	32	37	42	47	52	57	62	67	72	77	82	87	92	97	102	107	112	117	122	127	132	137	142	147	152	157
24	19	14	9	4	1	6	11	16	21	26	31	36	41	46	51	56	61	66	71	76	81	86	91	96	101	106	111	116	121	126	131	136	141	146	151
30	25	20	15	10	5	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125	130	135	140	145
36	31	26	21	16	11	6	1	4	9	14	19	24	29	34	39	44	49	54	59	64	69	74	79	84	89	94	99	104	109	114	119	124	129	134	139
42	37	32	27	22	17	12	7	2	3	8	13	18	23	28	33	38	43	48	53	58	63	68	73	78	83	88	93	98	103	108	113	118	123	128	133
48	43	38	33	28	23	18	13	8	3	2	7	12	17	22	27	32	37	42	47	52	57	62	67	72	77	82	87	92	97	102	107	112	117	122	127
54	49	44	39	34	29	24	19	14	9	4	1	6	11	16	21	26	31	36	41	46	51	56	61	66	71	76	81	86	91	96	101	106	111	116	121
60	55	50	45	40	35	30	25	20	15	10	5	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115
66	61	56	51	46	41	36	31	26	21	16	11	6	1	4	9	14	19	24	29	34	39	44	49	54	59	64	69	74	79	84	89	94	99	104	109
72	67	62	57	52	47	42	37	32	27	22	17	12	7	2	3	8	13	18	23	28	33	38	43	48	53	58	63	68	73	78	83	88	93	98	103

$$M^6_{max} = \begin{cases} \left\lfloor \frac{5}{2}pq \right\rfloor & \text{when q is even} \\ \left\lfloor \frac{5}{2}pq \right\rfloor - q & \text{when q is odd} \end{cases} \leq \left\lfloor \frac{5}{2} N_1 N_2 N_3 N_4 \right\rfloor$$

6th-Order: General Form

$$\mathbb{S} = \mathbb{P}_1 \cup \mathbb{P}_2 \cup \mathbb{P}_3 \cup \mathbb{Q}_1 \cup \mathbb{Q}_2 \cup \mathbb{Q}_3$$

Examine 9 Combinations of:

$$\begin{matrix} -\mathbb{P}_1 + \mathbb{P}_2 + \mathbb{P}_3 \\ +\mathbb{P}_1 - \mathbb{P}_2 + \mathbb{P}_3 \\ \mathbb{P}_1 + \mathbb{P}_2 - \mathbb{P}_3 \end{matrix} \quad \text{and} \quad \begin{matrix} -\mathbb{Q}_1 + \mathbb{Q}_2 + \mathbb{Q}_3 \\ +\mathbb{Q}_1 - \mathbb{Q}_2 + \mathbb{Q}_3 \\ +\mathbb{Q}_1 + \mathbb{Q}_2 - \mathbb{Q}_3 \end{matrix}$$

Nested

Nested

Coprime

6th-Order: General Form

$$\mathbb{S} = \mathbb{P}_1 \cup \mathbb{P}_2 \cup \mathbb{P}_3 \cup \mathbb{Q}_1 \cup \mathbb{Q}_2 \cup \mathbb{Q}_3$$

$$\mathbb{P}_1 = \{(n_1 N_2 N_3)p \mid n_1 = 0, 1, 2, \ldots, N_1\}$$
$$\mathbb{P}_2 = \{(n_2 N_3 + q)p \mid n_2 = 0, 1, 2, \ldots, N_2\}$$
$$\mathbb{P}_3 = \{(n_3 + \left\lfloor \frac{3q}{2} \right\rfloor)p \mid n_3 = 0, 1, 2, \ldots, N_3\}$$
$$\mathbb{Q}_1 = \{(n_4 N_5 N_6 - \left\lfloor \frac{5p}{2} \right\rfloor)q \mid n_4 = 0, 1, 2, \ldots, N_4\}$$
$$\mathbb{Q}_2 = \{(n_5 N_6 - \left\lfloor \frac{7p}{2} \right\rfloor)q \mid n_5 = 0, 1, 2, \ldots, N_5\}$$
$$\mathbb{Q}_3 = \{(n_6 - 5p)q \mid n_6 = 0, 1, 2, \ldots, N_6\}$$

6th-Order: Integration

12	17	22	27	32	37	42	47	52	57	62	67	72	77	82	87	92	97	102	107	112	117	122	127	132	137	142	147	152	157	162	167	172	177	182	187
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$$M^6_{max} = \left\lfloor \frac{17}{2} pq \right\rfloor \le \left\lfloor \frac{17}{2} N_1 N_2 N_3 N_4 N_5 N_6 \right\rfloor$$

2q-th Order: Layering

- 6-6-6-6…6
- 6-6-6-6…4
- 6-6-6-6…2

2q-th Order: Layering

$$\begin{matrix} \mathbb{S}_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_{N_1}\} \\ \mathbb{S}_2 = \{\beta_1, \beta_2, \ldots, \beta_{N_2}\} \end{matrix} \quad \text{with} \quad \begin{matrix} \Phi^{2q_1}(\mathbb{S}_1) = \{-\mu_1 \le \mu \le \mu_1\} \\ \Phi^{2q_2}(\mathbb{S}_2) = \{-\mu_2 \le \mu \le \mu_2\} \end{matrix}$$

Take a new $2(q_1 + q_2)$-th order ruler:

$$\mathbb{S}_1 \cup \mathbb{S}_2'$$

$$\mathbb{S}_2' = \{2\beta_1\mu_1, 2\beta_2\mu_1, \ldots, 2\beta_{N_2}\mu_1\}$$

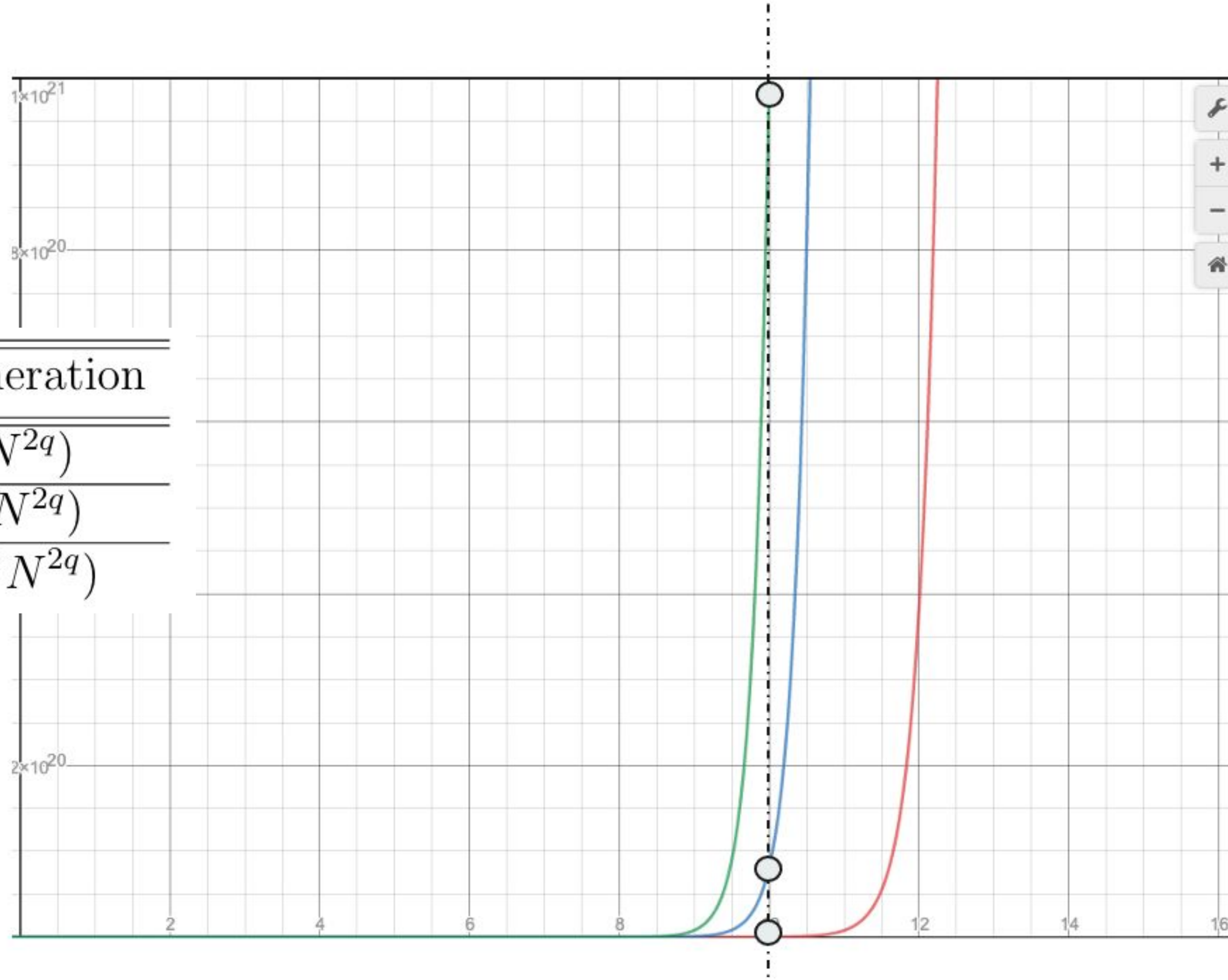
This generates:

$$\Phi^{2(q_1+q_2)}(\mathbb{S}_1 \cup \mathbb{S}_2') = \{-2\mu_1\mu_2 - \mu_1 \le \mu \le 2\mu_1\mu_2 + \mu_1\}$$

2q-th Order: Result

- 6-6-6-6…6 $O(17^{\frac{q}{3}} N^{2q})$
- 6-6-6-6…4 $O(2 \cdot 17^{\frac{q-1}{3}} N^{2q})$
- 6-6-6-6…2 $O(5 \cdot 17^{\frac{q-2}{3}} N^{2q})$

Ruler Structures	Lag Generation
$2q$L-NA	$O(2N^{2q})$
SE-$2q$L-NA	$O(2^q N^{2q})$
(proposed)	$O(17^{\frac{q}{3}} N^{2q})$



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Questions?

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THANK YOU!