The master field and free Brownian motions

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- Two random variables in $\mathcal{A}$ are said to be *free* if they satisfy a particular infinite set of relations involving φ. We should think of freeness as the non-commutative analogue to independence in classical probability theory.

Noncrossing partitions

A noncrossing partition of $\{1, \ldots, n\}$ is a partition in which no two blocks of the partition "cross" when drawn as shown.

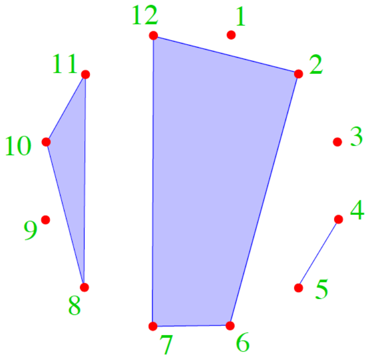

Figure: A noncrossing partition of $\{1, \ldots, 12\}$

Moment-cumulant relation

Theorem (Free cumulants)

For all $n \geq 1$*, we inductively define the cumulants* $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ *to be multilinear functionals obeying the moment-cumulant relation*

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi(a_1, \ldots, a_n)$$

for all $a_1, \ldots, a_n \in \mathcal{A}$*. The set* $NC(n)$ *consists of the noncrossing partitions on* $\{1, \ldots, n\}$*, and* κ_π *represents the product of the cumulants* $\kappa_{|\pi_i|}$*, where each* π_i *is a block in* π *with size* $|\pi_i|$*.*

Example of moment-cumulant relation

Example

When $n = 3$, the noncrossing partitions are $\{\{1\},\{2\},\{3\}\}, \{\{1,2\},\{3\}\}$, $\{\{2,3\},\{1\}\}$, $\{\{1,3\},\{2\}\}$, $\{\{1,2,3\}\}$. The moment-cumulant relation tells us

$$
\begin{aligned}
\varphi(a_1 a_2 a_3) &= \sum_{\pi \in \mathrm{NC}(3)} \kappa_\pi(a_1, a_2, a_3) \\
&= \kappa_1(a_1)\kappa_1(a_2)\kappa_1(a_3) + \kappa_2(a_1, a_2)\kappa_1(a_3) + \kappa_2(a_2, a_3)\kappa_1(a_1) \\
&\quad + \kappa_2(a_1, a_3)\kappa_1(a_2) + \kappa_3(a_1, a_2, a_3)
\end{aligned}
$$

Free cumulants

Cumulants possess the following very useful property.

Theorem (Mixed cumulants vanish)

The elements $x, y \in \mathcal{A}$ *are free if and only if* $\kappa_n(a_1, \ldots, a_n) = 0$ *whenever* $n \geq 2$*, all* a_i *are either* x *or* y*, and* $a_i \neq a_j$ *for some* i, j*. This result can be naturally extended for more than two elements.*

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Example

Suppose that $x, y \in \mathcal{A}$ are free and we wish to calculate $\varphi(xyx)$. By the moment-cumulant relation,

$$\varphi(xyx) = \kappa_1(x)\kappa_1(y)\kappa_1(x) + \kappa_2(x, y)\kappa_1(x) + \cdots + \kappa_3(x, y, x).$$

Since x, y are free, all mixed cumulants vanish. So, we are just left with

$$\varphi(xyx) = \kappa_1(x)\kappa_1(y)\kappa_1(x) + \kappa_2(x, x)\kappa_1(y).$$

Free Brownian motions

Theorem (Biane, 1997)

The free multiplicative Brownian motion is a collection $(u_t)_{t\geq 0}$ of unitary random variables ($u_t^* = u_t^{-1}$) in a non-commutative probability space $(\mathcal{A}, \varphi)$. Its distribution is characterized as follows.

- *For all $0 \leq s < t$, the element $u_t u_s^*$ has the same distribution as u_{t-s}.*
- *For all $0 \leq t_1 < \cdots < t_n$, the elements $u_{t_1}, u_{t_2}u_{t_1}^*, \ldots, u_{t_n}u_{t_{n-1}}^*$ form a free family.*
- *The moments are given by*

$$\varphi(u_t^n) = e^{-\frac{nt}{2}} \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} n^{k-1} \binom{n}{k+1}.$$

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- Any loop in $\mathrm{L}_o(\mathbb{R}^2)$ induces a graph $\mathbb{G}$ such that the loop can be written as a concatenation of edges of $\mathbb{G}$.
- For any planar graph $\mathbb{G}$, let $\mathrm{L}_o(\mathbb{G})$ be the set of loops on $\mathbb{G}$, i.e., loops formed by concatenating edges of $\mathbb{G}$.
- Define an equivalence relation on $\mathrm{L}_o(\mathbb{G})$ in which two loops are equivalent if one can be obtained from the other through a finite sequence of insertions and deletions of expressions of the form ee^{-1}, where e is an edge.

Loops on graphs

Example

Applying this natural "backtrack cancellation", the loops $l_1 = e_2 e_1 e_1^{-1} e_3$ and $l_2 = e_2 e_3 e_1 e_4^{-1} e_4 e_1^{-1}$ are both equivalent to the loop $l_3 = e_2 e_3$.

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- Let $\mathrm{RL}_o(\mathbb{G})$ be the quotient of $\mathrm{L}_o(\mathbb{G})$ by this equivalence relation. This is the space of *reduced loops* on $\mathbb{G}$ based at o.

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Theorem (Lévy, 2011)

The space $\mathrm{RL}_o(\mathbb{G})$ *is a free group with rank equal to the number of bounded faces in* $\mathbb{G}$*. Furthermore, this free group has many bases indexed by the set of bounded faces.*

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The space $\mathrm{RL}_o(\mathbb{G})$ is a free group with rank equal to the number of bounded faces in $\mathbb{G}$. Furthermore, this free group has many bases indexed by the set of bounded faces.

- One such basis is the *lasso basis*, which is a collection of loops in $\mathrm{RL}_o(\mathbb{G})$ determined by picking a spanning tree on $\mathbb{G}$.

The master field

Theorem (Lévy, 2011)

The master field is a collection $(h_l)_{l \in \mathsf{L}_o(\mathbb{R}^2)}$ of random variables in a non-commutative probability space $(\mathcal{A}, \tau)$, indexed by the loops in the plane. Its distribution is fully characterized by the following properties.

- *For all $l, l_1, l_2 \in \mathsf{L}_o(\mathbb{R}^2)$, the equalities $h_{l^{-1}} = h_l^{-1} = h_l^*$ and $h_{l_1 l_2} = h_{l_2} h_{l_1}$ hold.*
- *It is continuous in the space of loops, i.e., if the loops $(l_n)_{n \geq 0}$ converge to l, then $(h_{l_n})_{n \geq 0}$ converges in distribution to h_l.*
- *For any planar graph $\mathbb{G}$ in $\mathbb{R}^2$ and lasso basis $\{\lambda_F : F \in \mathbb{F}^b\}$ on $\mathbb{G}$, the finite collection $(h_{\lambda_F})_{F \in \mathbb{F}^b}$ is a collection of mutually free random variables such that for every $F \in \mathbb{F}^b$, the distribution of h_{λ_F} is a free Brownian motion stopped at time $|F|$.*

The master field

From the previous definition, we can think of the master field as a function $\Phi : \mathsf{L}_o(\mathbb{R}^2) \to \mathbb{C}$ defined by

$$\Phi(l) = \tau(h_l).$$

This is because we can write l as a product of lassos and then apply the "anti-multiplicativity" property.

Example of a master field calculation

Consider the loop l below, which has two bounded faces with area s and t.

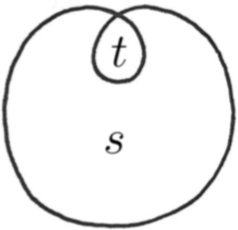

Figure: A loop with two bounded face

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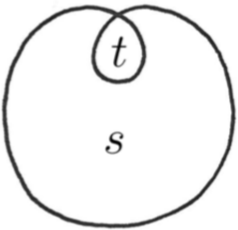

Figure: A loop with two bounded face

We can compute the master field of this loop to be

$$\Phi(l) = e^{-\frac{s}{2}-t}(1-t).$$

The Makeenko–Migdal equations

The Makeenko–Migdal equations give an efficient way to compute $\Phi(l)$ for any loop l through a system of differential equations. The main idea is to treat $\Phi(l)$ as a function of the areas of the bounded faces delimited by l.

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Theorem (Makeenko–Migdal)

Let l be a loop, and fix a point of self-intersection with exactly two ingoing strands and two outgoing strands. Let l_1 and l_2 be the two loops formed by swapping which outgoing strand connects to each ingoing strand. Label the four faces cyclically around the intersection $F_1, \ldots, F_4$ with F_1 adjacent to the two outgoing strands. Then, $\Phi(l)$ satisfies the equation

$$\left(\frac{d}{d|F_1|} - \frac{d}{d|F_2|} + \frac{d}{d|F_3|} - \frac{d}{d|F_4|}\right)\Phi(l) = \Phi(l_1)\Phi(l_2).$$

Example of the Makeenko–Migdal equations

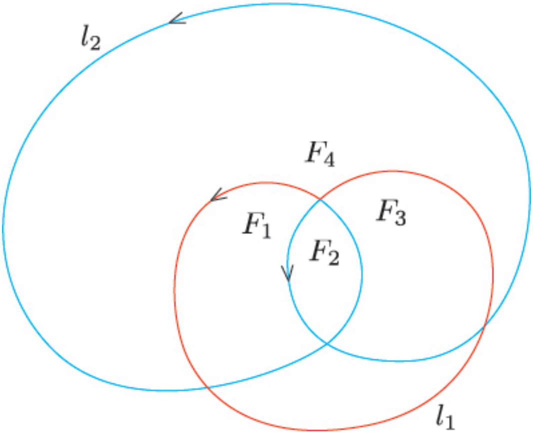

Figure: Setup for the Makeenko–Migdal equations

Results

- In literature, the master field is defined as the large N limit of the Yang–Mills holonomy process from two-dimensional Yang–Mills theory, which gives us many non-obvious properties for free when we pass to this limit.
- In our project, we redefined the master field as an object in its own right, independent from the finite N case.

Theorem

Under our definition, for any loop $l \in \mathsf{L}_o(\mathbb{R}^2)$, *the value of* $\Phi(l)$ *does not depend on the spanning tree chosen for the lasso basis.*

- We also discovered a different, more elementary proof of the Makeenko–Migdal equations.

Two-dimensional Yang–Mills theory

To specify a two-dimensional Yang–Mills theory, we need

- A compact surface Σ, which plays the role of space-time,
- A Lie group G, which describes the physical symmetries of the field and characterizes the particular kind of particle interaction,
- A principal G-bundle $\pi : P \to \Sigma$.

We want to construct and study a measure YM on the space of connections on P.

Motivation

This gives us a mathematically rigorous formulation for the *standard model* in two-dimensional Euclidean space-time.

Yang–Mills measure

- Instead of defining the Yang–Mills measure on the space of connections, we actually consider its image under the holonomy mapping.
- Given a connection on P, the holonomy is a multiplicative G-valued function on the space $\mathrm{L}_o(\Sigma)$ of loops on Σ based at some origin o.
- This holonomy mapping is injective and preserves symmetry, so we lose no information by defining the Yang–Mills measure on the image.
- Then, the Yang–Mills measure can be thought of as a collection $(H_l)_{l \in \mathrm{L}_o(\Sigma)}$ of G-valued random variables indexed by the set of loops. We call this the *Yang–Mills holonomy process*.

Yang–Mills holonomy process

The Yang–Mills holonomy process is a collection of G-valued random variables indexed by the set of loops in the plane based at some origin o.

Theorem (Yang–Mills holonomy process)

The distribution of the $(H_l)_{l \in \mathsf{L}_o(\mathbb{R}^2)}$ is fully characterized by the following.

- *For all $l, l_1, l_2 \in \mathsf{L}_o(\mathbb{R}^2)$, the equalities $H_{l^{-1}} = H_l^{-1}$ and $H_{l_1 l_2} = H_{l_2} H_{l_1}$ hold almost surely.*
- *It is stochastically continuous in the space of loops, i.e., if the loops $(l_n)_{n \geq 0}$ converge to l, then $(H_{l_n})_{n \geq 0}$ converges in probability to H_l.*
- *For any planar graph $\mathbb{G}$ in $\mathbb{R}^2$ and lasso basis $\{\lambda_F : F \in \mathbb{F}^b\}$ on $\mathbb{G}$, the distribution of $(H_l)_{l \in \mathsf{L}_o(\mathbb{G})}$ is fully characterized by the distribution of the finite collection $(H_{\lambda_F})_{F \in \mathbb{F}^b}$. This is a collection of independent random variables such that for every $F \in \mathbb{F}^b$, the distribution of H_{λ_F} is a Brownian motion on G stopped at time $|F|$.*

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Image credits

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