Factorization Invariants of Algebraic Valuations of Positive Cyclic Semirings

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Monoids

Definition (monoid)

A set M with an (additive) associative (and commutative) operation $+$ is called a ***monoid*** if M is closed under $+$ and contains an element 0 such that $0 + x = x$ for all $x \in M$.

Examples

1. $(\mathbb{N}_0, +)$, where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, and $(\mathbb{R}_{\geq 0}, +)$ are monoids.
2. $(\mathbb{Z}, +)$ and $(\mathbb{R}, +)$ are monoids.
3. In general, any (abelian) group is a monoid.
4. $(\{0\} \cup \{2, 3, 4, \dots\}, +) = (\mathbb{N}_0 \setminus \{1\}, +)$ is a monoid.
5. $\left(\left\{\frac{m}{2^n} : m, n \in \mathbb{N}_0\right\}, +\right)$ is a monoid.

Atomic Monoids

Let $(M, +)$ be a monoid.

Definition (units, atoms)

An element $x \in M$ is a ***unit*** if there exists $y \in M$ such that $x + y = 0$. An ***atom*** $a \in M$ is a non-unit element such that if $a = x + y$ for some $x, y \in M$, then x or y is a unit. Let $\mathcal{A}(M)$ denote the set of atoms in M.

Definition (atomic monoid)

Monoid M is ***atomic*** if every non-unit element in M is the sum of atoms.

The notion of atomicity was first studied by Paul Cohn back in 1968 in the context of commutative algebra. It has been studied systematically since then in both commutative rings and commutative monoids.

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Definition (atomic monoid)

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Examples

1. $\mathbb{N}_0$ is atomic with $\mathcal{A}(\mathbb{N}_0) = \{1\}$.
2. $M = \{0\} \cup \mathbb{R}_{\geq 1}$ is atomic with $\mathcal{A}(M) = [1, 2)$.
3. $\mathbb{R}_{\geq 0}$ is NOT atomic. Indeed, $\mathcal{A}(\mathbb{R}_{\geq 0}) = \varnothing$.

Factorizations

Definition (factorization, length)

Let x be a non-unit element of M. A decomposition $x = a_1 + a_2 + \cdots + a_\ell$, where $a_1, a_2, \ldots, a_\ell \in \mathcal{A}(M)$ is called a ***factorization*** of x with ***length*** ℓ.

Then we set
$$\mathsf{L}(x) := \{\ell \in \mathbb{N} : \ell \text{ is a length of } x\}.$$

The interest in the study of factorization theory comes from algebraic number theory, as the rings of integers of many algebraic number fields are NOT unique factorization domains (UFDs).

For instance, $\mathbb{Z}[\sqrt{-5}]$ is not a UFD as

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

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Examples

1. Every element in $(\mathbb{N}_0, +)$ has exactly one factorization. For instance, $3 = 1 + 1 + 1$ is a factorization of 3 of length 3. In general, the only factorization of $m \in \mathbb{N}_0$ has length m.
2. If $M = \{0\} \cup \mathbb{N}_{\geq 2}$, then $\mathcal{A}(M) = \{2, 3\}$. Observe that $6 = 2 + 2 + 2 = 3 + 3$, which means $\mathsf{L}(6) = \{2, 3\}$.

Cyclic Semiring

Let $\mathbb{N}_0[x]$ denote the set of polynomials with coefficients in $\mathbb{N}_0$.

Remark: For all $\alpha \in \mathbb{R}_{>0}$, the set $\{p(\alpha) : p(x) \in \mathbb{N}_0[x]\}$ is closed under both addition and multiplication.

Definition (cyclic semiring)

For each positive algebraic number α, we let $\mathbb{N}_0[\alpha]$ denote the set $\{p(\alpha) : p(x) \in \mathbb{N}_0[x]\}$. We call $(\mathbb{N}_0[\alpha], +)$ the ***cyclic semiring*** of α.

Cyclic semirings naturally show up as positive valuations of certain commutative rings in information theory, and the study of factorizations in these semirings was initiated by Chapman et al.

Examples and PRIMES Project Purpose

Examples

1. If $\alpha = 1$, then $\mathbb{N}_0[\alpha] = \mathbb{N}_0[1] = \mathbb{N}_0$. Then $\mathbb{N}_0[\alpha]$ is atomic and $\mathcal{A}(\mathbb{N}_0[1]) = \{1\}$.
2. If $\alpha = \sqrt{2}$, then $\mathbb{N}_0[\alpha] = \mathbb{N}_0[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{N}_0\}$. Then $\mathbb{N}_0[\sqrt{2}]$ is atomic and $\mathcal{A}(\mathbb{N}_0[\sqrt{2}]) = \{1, \sqrt{2}\}$.
3. If $\alpha = 1/n$ for some $n \in \mathbb{Z}_{\geq 2}$, then $\mathcal{A}(\mathbb{N}_0[1/n]) = \varnothing$. This is because $(1/n)^k$ can be expressed as the sum of n copies of $(1/n)^{k+1}$, and so $(1/n)^k$ is not an atom for any $k \in \mathbb{N}_0$. Hence $\mathbb{N}_0[1/n]$ is not atomic.

Our PRIMES Project studies the factorizations and atoms of $\mathbb{N}_0[\alpha]$:

- How far elements in $\mathbb{N}_0[\alpha]$ are from having unique factorization?
- How far the atoms of $\mathbb{N}_0[\alpha]$ are from being primes?

To answer these questions, we study two of the most relevant atomic statistics: the **elasticity** and the **omega–primality**.

Elasticity

Definition (elasticity)

For an atomic monoid M, the ***elasticity*** of non-unit $x \in M$ is defined as

$$\rho(x) := \frac{\sup \mathsf{L}(x)}{\inf \mathsf{L}(x)}.$$

In addition, the ***elasticity*** of the whole monoid M is defined as

$$\rho(M) \;=\; \sup\{\rho(x) : x \in M \text{ is a non-unit element}\}.$$

Remark

- $\rho(x) \in \mathbb{Q}_{\geq 1} \cup \{\infty\}$ for all non-unit x, and so $\rho(M) \in \mathbb{R}_{\geq 1} \cup \{\infty\}$.
- $\rho(x) = 1$ if and only if $\mathsf{L}(x)$ has size 1.

The elasticity was introduced in the 1980s by Valenza to measure how far the ring of integers of an algebraic number field is from being a UFD. It has been systematically studied since then.

Elasticity of Algebraic Cyclic Semirings I

Theorem (Jiang, Li, Zhu, 2021)

For any algebraic number $\alpha \in \mathbb{R}_{>0}$, the following statements hold.

1. *If $\alpha \in \mathbb{Q} \setminus (\{1/n\} \cup \mathbb{N}_0)$, then*
   - $\mathcal{A}(\mathbb{N}_0[\alpha]) = \{\alpha^n \mid n \in \mathbb{N}_0\}$ *and*
   - $\rho(\mathbb{N}_0[\alpha]) = \infty$.
2. *If $\alpha \notin \mathbb{Q}$ and $|\mathcal{A}(\mathbb{N}_0[\alpha])| = \infty$, then*
   - $\mathcal{A}(\mathbb{N}_0[\alpha]) = \{\alpha^n \mid n \in \mathbb{N}_0\}$ *and*
   - $\rho(\mathbb{N}_0[\alpha]) = \infty$.

Elasticity of Algebraic Cyclic Semirings II

Theorem (Jiang, Li, Zhu, 2021)

Let $\alpha \in \mathbb{R}_{>0}$ be an algebraic number whose minimal polynomial is of the form

$$m_\alpha(X) \; = \; X^n + \sum_{i=k}^{n-1} a_i X^i - \sum_{j=0}^{k-1} a_j X^j,$$

where $a_0, a_1, \ldots, a_{n-1} \in \mathbb{N}_0$ and $k \in \{1, 2, \ldots, n-1\}$. If $|\mathcal{A}(\mathbb{N}_0[\alpha])| = n + 2$, then

$$\rho(\mathbb{N}_0[\alpha]) = \frac{a_{k-1}^2 + (a_k + a_{k-1}) \left(\sum_{i=0}^{k-2} a_i\right)}{a_k^2 + (a_k + a_{k-1}) \left(1 + \sum_{j=k-1}^{n-1} a_j\right)}.$$

Elasticity of Algebraic Cyclic Semirings III

Theorem (Jiang, Li, Zhu, 2021)

Let $\alpha \in \mathbb{R}_{>0}$ be an algebraic number whose minimal polynomial is $m_\alpha(X) = X^3 - pX^2 + qX - r$, where $p, q, r \in \mathbb{N}_0$. If $|\mathcal{A}(\mathbb{N}_0[\alpha])| = 5$, then

$$\rho(\mathbb{N}_0[\alpha]) = \begin{cases} \frac{r^2-r+pq}{q^2+q-pr}, & \text{if } q^2 > pr \text{ and } q \geq p, \\ \frac{p^2-q+qr}{q^2+p-pr}, & \text{if } q^2 > pr \text{ and } q < p, \\ \frac{p^2-q+qr}{q^2+p-pr}, & \text{if } q^2 \leq pr. \end{cases}$$

Omega–Primality

Definition: Let M be a monoid. For $b, c \in M$, we say b ***divides*** c and write $b \mid_M c$ if $c = a + b$ for some $a \in M$. A non-unit $p \in M$ is ***prime*** if for all $a, b \in M$, $p \mid_M a + b$ implies $p \mid_M a$ or $p \mid_M b$.

Definition (omega–primality)

Let M be an atomic monoid. For $a \in \mathcal{A}(M)$, let $\omega(a)$ denote the smallest $\mathbb{N} \cup \{\infty\}$ such that if $a \mid_M a_1 + a_2 + \cdots + a_n$ for $a_1, a_2, \ldots, a_n \in \mathcal{A}(M)$, then there exists $J \subseteq \{1, 2, \ldots, n\}$ with $|J| \le \omega(a)$ such that $a \mid_M \sum_{j \in J} a_j$. We call $\omega(a)$ the ***omega–primality*** of a. We set

$$\omega(M) := \sup\{\omega(a) : a \in \mathcal{A}(M)\}.$$

Remark

- If a is an atom of M, then $\omega(a) = 1$ if and only if a is prime.
- The omega–primality measures how far an atom is from being prime.

Omega–Primality

Theorem (Jiang, Li, Zhu, 2021)

Let $\alpha \in \mathbb{R}_{>0}$ *be an algebraic number such that* $0 < \alpha < 1$ *and* $\mathbb{N}_0[\alpha]$ *is atomic. Then the following statements hold.*

1. $\omega(a) = \infty$ *for all* $a \in \mathcal{A}(\mathbb{N}_0[\alpha])$*, and so*
2. $\omega(\mathbb{N}_0[\alpha]) = \infty$.

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