# Square Tilings of Translation Surfaces

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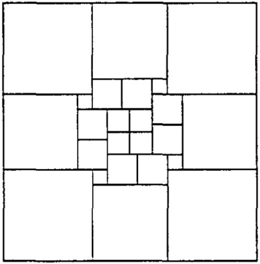
# An Introduction: Square Tilings

**Definition**

A *square tiling* of $\mathcal{P}$ is a set of non-overlapping squares which cover $\mathcal{P}$ and which are all contained by $\mathcal{P}$.

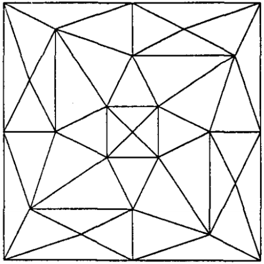


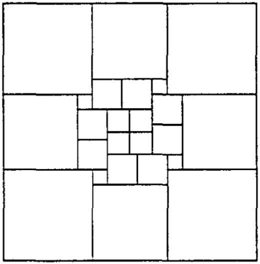
An example of a square tiling

# How to Analyze Tilings? The Contacts Graph

## Definition (Contacts Graph)

- Graph $G$
- Vertices $S$ correspond to squares $Z_S$
- Edge between $U$ and $V$ if and only if squares $Z_U$ and $Z_V$ share a side





A square tiling (right) with its contacts graph (left)

## Contacts Graphs and Triangulations, Part I

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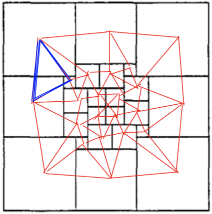
**Lemma**

*The contacts graph is always a triangulation.*



# Contacts Graphs and Triangulations, Part II

### Main Questions

- Given a triangulation, is there always a tiling with it as contacts graph?
- Is it unique?
- Can we construct such a tiling?

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### Answer [Schramm 1993]

Yes!

### Main Proof Idea:

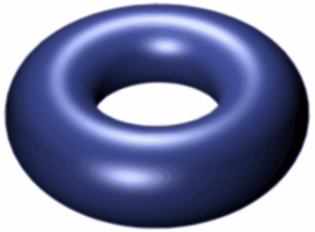
- Extremal Metric!

# From Squares to Translation Surfaces: Step I, the Torus!

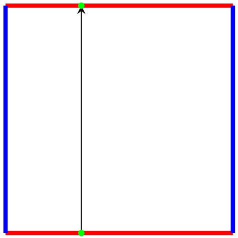
**Definition**

The *torus*:

- donut
- square with opposite sides identified as the same

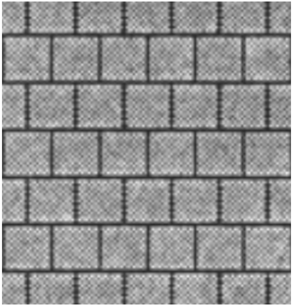


Torus



Torus as a square with identified edges.

# Example Tiling of a Torus



Doubly Periodic / Torus Tiling

# Theorems on the Torus

Theorem (Schramm 1996, C. 2020)

- *For any triangulation, a square tiling with it as contacts graph always exists! [Schramm]*
- *It is unique up to horizontally translating each square, or vertically translating each square. We can construct it if we know the square sizes. [Our result!]*

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### Proof Sketch

- Existence: proved by Oded Schramm via conformal geometry methods
- Uniqueness and Construction Method: obtained by **adapting the extremal metric**

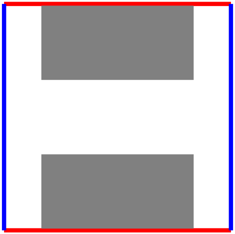
# Translation Surfaces (Finally!)

## Definition (Translation Surface)

- Take polygon with pairs of parallel sides
- Identify the opposite sides

## Example

The torus!



A square on a torus.

# The Generalized Problems: Tiling on Translation Surfaces!

**Questions**

- When can we tile a translation surface with squares?
- Does every triangulation correspond to a tiling?
- Is the tiling unique?
- How can we construct the tiling?

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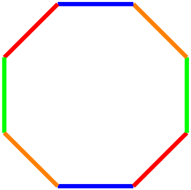
Our general problem is very difficult, so let's look at a particular case!

# Octagonal Translation Surfaces

- An octagonal translation surface is formed by identifying opposite edges of an octagon with four pairs of parallel sides.
- The surface encloses a region of genus 2.



# Square Tiling Octagonal Translation Surfaces, Part I

Theorem (C. 2020)

- *We cannot square tile the translation surface generated by the regular octagon.*
- *There exists a vertical stretch of the regular octagon, $R$, such that the translation surface generated by $R$ is square tileable.*

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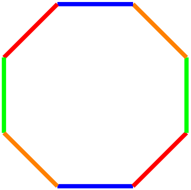
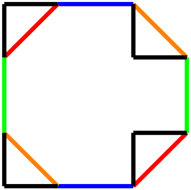
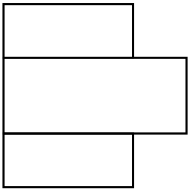


Regular Octagon
Translation Surface



Moving Triangles



Rectangle Tiling

# Square Tiling Octagonal Translation Surfaces, Part II

Proof Ideas:

- Define a notion of special side length and area
- Length is an additive function
- Area is product of width and height
- Choose function so the total area is negative

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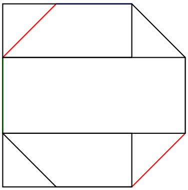
Proof Ideas:

- Define a notion of special side length and area
- Length is an additive function
- Area is product of width and height
- Choose function so the total area is negative
- Tile the octagonal surface with rectangles and then apply a vertical stretch



Three rectangles tiling the regular octagonal translation surface. Applying a vertical stretch with a factor of $1+\sqrt{2}$ makes them squares.

# Summary and Future Work

Our work:

- Proof that the square tiling corresponding to any triangulation
- Algorithm to construct such tilings
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Natural continuations:

- Triangulations/contacts graphs
- Other translation surfaces
- Adapt metric idea

# Acknowledgements


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