The Gauss-Epple homomorphism(s)

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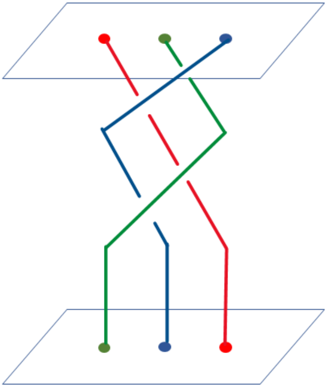
Overview

(For more details, see our paper, which will be out soon.)

Braids

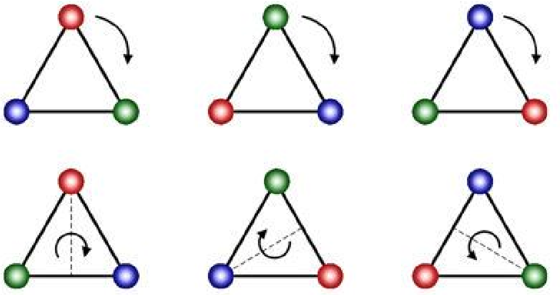
A braid on n strands is an arrangement of n strings that cannot pass through each other with both endpoints fixed up to isotopy. We number the strands in order.



An example braid on three strands.

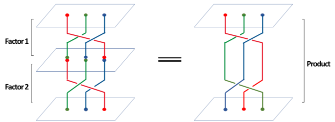
Groups

A group is a set with a binary operation that is associative, has an identity, and is invertible.

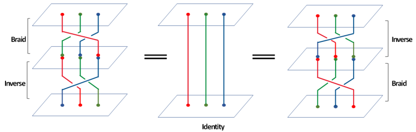


The elements of the group $D_6 \simeq S_3$.

The braid group



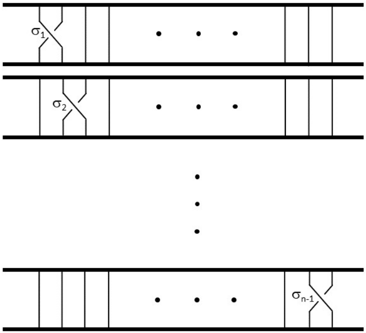
Braid multiplication.



A braid being composed with its inverse to form the identity braid.

As shown in the diagram, we can attach the ends of one braid to the beginnings of another. This is an associative operation. Moreover, braids can also be inverted, and an identity braid exists. Therefore, the set of all braids on n strands up to isotopy forms a group, the braid group B_n.

Artin generators



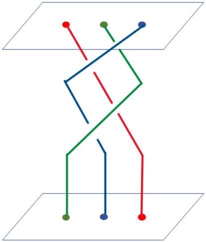
Artin generators.

The Artin generators σ_i are the braids where the i^{th} strand crosses once over the $(i+1)^{\text{th}}$ strand. Every braid can be decomposed into Artin generators. We have

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \ \forall i; \ \sigma_i \sigma_j = \sigma_j \sigma_i \ \forall i,j : |i-j| \geq 2 \rangle.$$

Writhe and permutation

The permutation is the homomorphism $B_n \to S_n$ that remembers what the order of the strands is at the bottom as a permutation of the order of the strands at the top.



$(\sigma_2^{-1}\sigma_1^{-1})^2$, which has permutation (1,3,2) and writhe -4.

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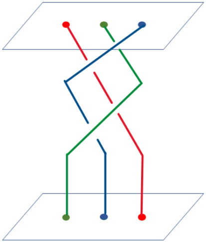

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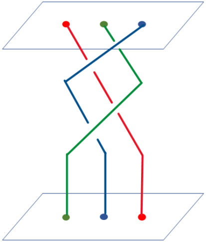
We combine the two previously mentioned into one homomorphism $WP : B_n \to \mathbb{Z} \times S_n$, which we will call the writhe-permutation homomorphism.



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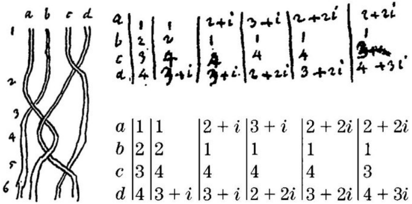
The Gauss-Epple homomorphism?



a	1	1	$2+i$	$3+i$	$2+2i$	$2+2i$
b	2	2	1	1	1	1
c	3	4	4	4	4	3
d	4	$3+i$	$3+i$	$2+2i$	$3+2i$	$4+3i$

Gauss's note.

In a note written by Gauss, the braid $\sigma_3\sigma_1\sigma_2^{-2}\sigma_3$, with strands labeled a, b, c, d from left to right, is depicted with a corresponding table. Epple noticed that this table could be about a braid invariant.

Formal definition

As defined by Epple, the Gauss-Epple homomorphism is a group action α of B_n on the Gaussian integers $\mathbb{Z}[i]$ defined by the generating relation

$$\alpha(\sigma_k)(z) := \begin{cases} z & \text{if } \mathrm{Re}(z) \neq k, k+1 \\ z+1 & \text{if } \mathrm{Re}(z) = k \\ z-1+i & \text{if } \mathrm{Re}(z) = k+1 \end{cases}$$

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We make several simplifications: We replace $\mathbb{Z}[i]$ with $\mathbb{Z}^2$, and then restrict to the induced action on the subset $[n] \times \mathbb{Z}$. We will refer to this homomorphism, thus modified, as GE.

The structure theorem

Here is our first result. In our notation, we have:

$$GE(\sigma_k)(a,b) := \begin{cases} (a,b) & \text{if } a \neq k, k+1 \\ (k+1,b) & \text{if } a = k \\ (k,b+1) & \text{if } a = k+1 \end{cases} .$$

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Note that the operation $GE(\sigma_k)$ permutes the value of a and adds a constant to b only depending on a; such operations are closed under composition. We abbreviate the map $[n] \times \mathbb{Z} \to [n] \times \mathbb{Z} : (a,b) \mapsto (\pi(a), a + \ell_b)$ as just $(\pi, \ell) \in S_n \ltimes \mathbb{Z}^n$. We infer the following:

Theorem (Structure theorem)

The Gauss-Epple homomorphism factors through the group $\mathbb{Z}^n \rtimes S_n$

Random braids in kerGE

We automatically generated random words of $\{\sigma_1, \ldots, \sigma_{n-1}, \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}\}$ of various lengths and checked whether the corresponding braids belonged to ker GE. This way, we discovered several nontrivial braids inside, such as $\sigma_1^{-1}\sigma_3^{-1}\sigma_2^2\sigma_3^{-1}\sigma_1^{-1}\sigma_2^2$ and $(\sigma_1\sigma_2^{-1})^3$.

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imGE, imWP

As another way to study ker GE, we also studied im GE, which is a supergroup of im WP. We studied im WP first because it's simpler (and because ker $GE \subset$ ker WP):

Theorem

The group im WP is a degree 2 subgroup of $\mathbb{Z} \times S_n$: namely, the subset of pairs $(\pi, w) \in S_n \times \mathbb{Z}$ where π is an odd permutation iff w is an odd integer.

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Later, we solved the full problem:

Theorem

The group im GE is a degree 2 subgroup of $\mathbb{Z}^n \rtimes S_n$: namely, the subset of pairs $(\pi, \ell) \in S_n \times \mathbb{Z}^n$ where π is an odd permutation iff the sum of the components of ℓ is odd.

The symmetric Gauss-Epple homomorphism

In his original paper, Epple also defined a "symmetric" version, the symmetric Gauss-Epple homomorphism, by the relation

$$\beta(\sigma_k)(z) := \begin{cases} z & \text{if } \mathrm{Re}(z) \neq k, k+1 \\ z+1+i & \text{if } \mathrm{Re}(z) = k \\ z-1+i & \text{if } \mathrm{Re}(z) = k+1 \end{cases}$$

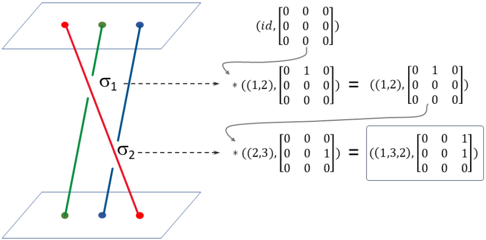
We also make the same simplifications as earlier and call the resulting object $symGE$. We proved that $\ker symGE = \ker GE$ exactly.

The super-Gauss-Epple homomorphism

We discovered the super-Gauss-Epple homomorphism, a homomorphism $superGE : B_n \to M_n(\mathbb{Z}) \rtimes S_n$ defined by

$$\sigma_k \mapsto ((k, k+1), e_k e'_{k+1}).$$

It has an even smaller kernel than GE.



An example calculation of superGE.

A 1-cocycle

We discovered the following function while investigating the image of *superGE*. Define the function $F : B_n \to M_n(\mathbb{Z})$ by the equation $superGE(\beta) = (\pi, M) \implies F(\beta) = M - M^T$. Then, for any left inverse $f : S_n \to B_n$ of the quotient map $B_n \to S_n$, we define $\overline{F}(\pi) = F(f(\pi))$.

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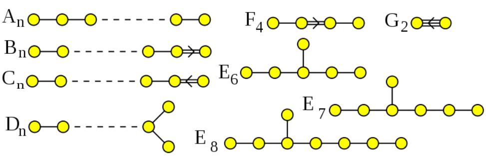
Theorem

- $\overline{F}$ *is well-defined and independent of the section* f.
- $\overline{F}$ *is a 1-cocycle: i.e. it satisfies the equation* $\overline{F}(\pi_1\pi_2) = \overline{F}(\pi_1) + \pi_1 * \overline{F}(\pi_2)$ *(where multiplication is by independently permuting rows and columns).*
- *For all permutations* π*, the upper half of* $\overline{F}(\pi)$ *has entries that are either zero or one.*

Artin and Coxeter groups

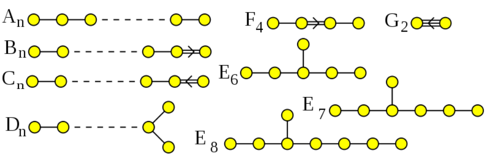
An Artin group
is a group presented by
a finite set of generators
and at most one braid
relation (i.e. a relation of
the form $a = b$, $ab = ba$,
$aba = bab$, $abab = baba$,
etc.) between
any two generators. The
presentation of an Artin
group can be depicted in a Dynkin diagram.



The Dynkin diagrams of finite Coxeter groups.

Artin and Coxeter groups

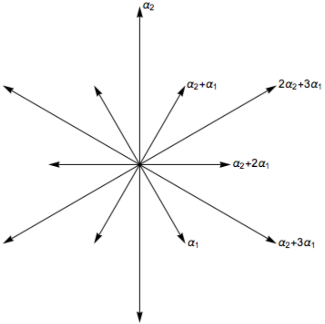
An Artin group is a group presented by a finite set of generators and at most one braid relation (i.e. a relation of the form $a = b$, $ab = ba$, $aba = bab$, $abab = baba$, etc.) between any two generators. The presentation of an Artin group can be depicted in a Dynkin diagram. A Coxeter group is a group presented with the generators and relations of an Artin group and the additional relations that the square of any canonical generator is the identity. A Coxeter group's presentation can also be depicted in a Dynkin diagram.



The Dynkin diagrams of finite Coxeter groups.

Root systems

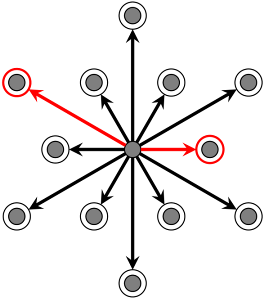
When a Coxeter group $\mathcal{C}$ is finite, it is also the group generated by the reflections upon a root system Φ, which is a special set of vectors. The associated Artin group $\mathcal{A}$ is then called an Artin group of finite type.



The G_2 root system.

Generalized super-Gauss-Epple homomorphisms

Pick some simple roots $\Delta \subset \Phi$.
We can associate each canonical Artin generator a with a simple root Δ_a.



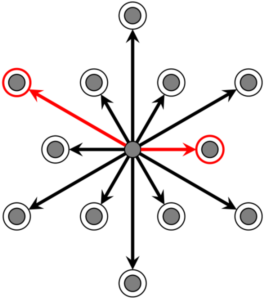
The G_2 root system with two simple roots highlighted.

Generalized super-Gauss-Epple homomorphisms

Pick some simple roots $\Delta \subset \Phi$.
We can associate each canonical Artin generator a with a simple root Δ_a.
Then we can define a homomorphism

$$\mathcal{A} \to \mathcal{C} \ltimes \mathbb{Z}^\Phi$$

by the relation $a \mapsto (\mathcal{C}(a), \Phi_a)$.
We also call this *superGE*.
Note that when $\mathcal{A} = B_n, \mathcal{C} = S_n, \Phi = \{e_i - e_j | i \neq j; 1 \leq i, j \leq n\}$, this gives the ordinary super-Gauss-Epple homomorphism.



The G_2 root system with two simple roots highlighted.

More 1-cocycles

Let $F : \mathcal{A} \to \mathbb{Z}^{\Phi}$ be defined by $superGE(a) = (c, v) \implies F(a)[\phi] = v_{\phi} - v_{-\phi}$. Then F factors through $\mathcal{C}$, with one factor being the map $\mathcal{A} \to \mathcal{C}$ and the other factor being a 1-cocycle of $\mathcal{C}$ on $\mathbb{Z}^{\Phi}$ that we also call $\overline{F}$.

Summary of results

We obtained the following results:

- A structure theorem for im GE.
- Asymptotic probability of random braids in ker GE.
- Construction of *superGE*.
- Discovery of a 1-cocycle.
- Generalizations to Artin groups of finite type (but not to complex reflection groups).

We suggest generalizations to complex reflection groups and homomorphisms $\mathcal{A} \to \mathcal{C} \ltimes \mathbb{Z}^X$ as areas for future work.

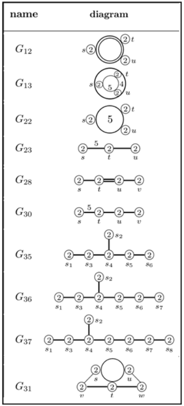


Table of some complex reflection groups.

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- Dr. Minh-Tam Trinh (advisor)
- MIT PRIMES program

Questions?